8.1 (a) Let $c \in \mathbb{R}$. Then, for $x \neq c$,

$$\frac{f(x)-f(c)}{x-c} = \frac{x^2-c^2}{x-c} = \frac{(x-c)(x+c)}{x-c} = x+c.$$

Hence,

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c}(x+c) = 2c.$$

So the function $f$ is differentiable at $c$. As $c$ was arbitrarily chosen, $f$ is differentiable everywhere and $f'(x) = 2x$.

As $f(3) = 9$ and $f'(3) = 6$, the tangent has slope 6 and contains the point $(3, 9)$. An equation of the tangent is $y - 9 = 6(x-3) \Longleftrightarrow y = 6x - 9$.

(c) Let $c \in \mathbb{R}$. Then, for $x \neq c$,

$$\frac{k(x)-k(c)}{x-c} = \frac{a-a}{x-c} = 0.$$

Hence,

$$\lim_{x\to c} \frac{k(x)-k(c)}{x-c} = 0.$$

So the function $k$ is differentiable at $c$. As $c$ was arbitrarily chosen, $k$ is differentiable everywhere and $k'(x) = 0$.

As $k(3) = a$ and $k'(3) = 0$, the tangent has slope 0 and contains the point $(3, a)$. An equation of the tangent is $y = a$.

(d) Let $c \neq 0$. Then, for $u \neq c$ and $u \neq 0$,

$$\frac{h(u)-h(c)}{y-c} = \frac{\frac{2}{3u} + A - \frac{2}{3c} - A}{u-c} = \frac{1}{u-c}\cdot\frac{2c-2u}{3uc} = -\frac{2}{3uc}.$$

Hence,

$$\lim_{u\to c} \frac{h(u)-h(c)}{u-c} = \lim_{u\to c} -\frac{2}{3uc} = -\frac{2}{3c^2}.$$

So the function $h$ is differentiable at $c$. As $c \neq 0$ was arbitrarily chosen, $h$ is differentiable everywhere (on its domain) and $h'(u) = -\dfrac{2}{3u^2}$.

As $h(3) = \frac{2}{9} + A$ and $h'(3) = -\frac{2}{27}$, the tangent has slope $-\frac{2}{27}$ and contains the point $(3, \frac{2}{9} + A)$. An equation of the tangent is $y - \frac{2}{9} - A = -\frac{2}{27}(x-3) \Longleftrightarrow y = -\frac{2}{27}x + \frac{4}{9} + A$.

8.3 Let $c > 0$. Note that for $x > 0$ and $x \neq c$,

$$\frac{f(x)-f(c)}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c} = \frac{\sqrt{x}-\sqrt{c}}{x-c}\,\frac{\sqrt{x}+\sqrt{c}}{\sqrt{x}+\sqrt{c}} = \frac{x-c}{(x-c)(\sqrt{x}+\sqrt{c})} = \frac{1}{\sqrt{x}+\sqrt{c}}.$$

So

$$\lim_{x\to c} \frac{f(x)-f(c)}{x-c} = \lim_{x\to c} \frac{1}{\sqrt{x}+\sqrt{c}} = \frac{1}{2\sqrt{c}}.$$

Hence, the function $f$ is differentiable at $c$ and $f'(c) = \dfrac{1}{2\sqrt{c}}$.

8.7 (a) For $x > 0$,

$$f'(x) = 2x\ln x + x^2 \cdot \frac{1}{x} + \ln x + x \cdot \frac{1}{x} = x + 1 + (2x+1)\ln x.$$

(b) For $x > 0$,

$$f'(x) = 2x\sqrt{x} + x^2 \cdot \frac{1}{2\sqrt{x}} = 2x\sqrt{x} + \tfrac{1}{2}x\sqrt{x} = 2\tfrac{1}{2}x\sqrt{x}.$$

(c) For any $x$,

$$f'(x) = -\cos x\, \mathrm{e}^x + (1 - \sin x)\mathrm{e}^x = (1 - \sin x - \cos x)\mathrm{e}^x.$$

(d) For any $x$,

$$f'(x) = \cos x \cdot \cos x + \sin x \cdot (-\sin x) = \cos^2 x - \sin^2 x.$$

8.8 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: the function $f^n$ is differentiable and $\big(f^n\big)'(x) = n\big(f(x)\big)^{n-1} f'(x)$ for every $x \in I$.

(1) The statement $\mathcal{P}(1)$ is true.

(2) Let $k \in \mathbb{N}$ and assume that the statement $\mathcal{P}(k)$ is true.

Then, according to the Product Rule, for $x \in I$

$$\begin{aligned}\big(f^{k+1}\big)'(x) &= \big(f \cdot f^k\big)'(x) = f'(x) \cdot f^k(x) + f(x) \cdot \big(f^k\big)'(x) \\ &= f'(x) \cdot f^k(x) + f(x) \cdot k\big(f(x)\big)^{k-1} f'(x) = (k+1)\big(f(x)\big)^k f'(x).\end{aligned}$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

8.10 (a) For $x > 0$,

$$f'(x) = \frac{\big(1 + \sqrt{x}\big) \cdot \dfrac{1}{\sqrt{x}} - 2\sqrt{x} \cdot \dfrac{1}{2\sqrt{x}}}{\big(1 + \sqrt{x}\big)^2} = \frac{\dfrac{1}{\sqrt{x}} + 1 - 1}{\big(1 + \sqrt{x}\big)^2} = \frac{1}{\sqrt{x}\big(1 + \sqrt{x}\big)^2}.$$

(b) For any $x$,

$$f'(x) = \frac{\big(x^2 + 1\big) \cdot 2x - (x^2 - 1) \cdot 2x}{\big(x^2 + 1\big)^2} = \frac{2x^3 + 2x - 2x^3 + 2x}{\big(x^2 + 1\big)^2} = \frac{4x}{\big(x^2 + 1\big)^2}.$$

(c) For any $x$,

$$\begin{aligned}f'(x) &= \frac{(2 + \cos x) \cdot -\cos x - (1 - \sin x) \cdot -\sin x}{(2 + \cos x)^2} = \frac{-2\cos x - \cos^2 x + \sin x - \sin^2 x}{(2 + \cos x)^2} \\ &= \frac{-2\cos x + \sin x - 1}{(2 + \cos x)^2}.\end{aligned}$$

(d) For $x > 0$,

$$f'(x) = \frac{\mathrm{e}^x \cdot \dfrac{1}{x} - \ln x \cdot \mathrm{e}^x}{\big(\mathrm{e}^x\big)^2} = \frac{\dfrac{1}{x} - \ln x}{\mathrm{e}^x} = \frac{1 - x\ln x}{x\,\mathrm{e}^x}.$$

8.13 (a)

$$f'(x) = g'\big(x + g(1)\big) \cdot 1 = g'\big(x + g(1)\big).$$

(b)

$$f'(x) = g'\big(x \cdot g(1)\big) \cdot g(1).$$

(c)

$$f'(x) = g'\big(x + g(x)\big) \cdot \big(1 + g'(x)\big).$$

(d) As $f(x) = g\big((x+1)^2\big)$,

$$f'(x) = g'\big((x+1)^2\big) \cdot 2(x+1).$$

8.28 According to the Chain Rule for every $x$

$$(f \circ g)'(x) = (g \circ f)'(x) \Longrightarrow f'(g(x)) \cdot g'(x) = g'(f(x)) \cdot f'(x) \Longrightarrow f'(x^2) \cdot 2x = 2f(x) \cdot f'(x).$$

In particular, for $x = 1$,

$$f'(1) = f(1) \cdot f'(1) \Longrightarrow f'(1)[1 - f(1)] = 0 \Longrightarrow f(1) = 1 \quad \text{or} \quad f'(1) = 0.$$

8.29 The function $x \mapsto x$ is differentiable on the interval $(1, \infty)$. According to the Arithmetic Rules for differentiable functions, also the function $x \mapsto 4x$ is differentiable on the interval $(1, \infty)$.

The functions $x \mapsto x^2$ and $x \mapsto 2$ are differentiable on the interval $(-\infty, 1)$. According to the Arithmetic Rules for differentiable functions, also the function $x \mapsto 2x^2 + 2$ is differentiable on the interval $(-\infty, 1)$.

Note that for $x > 1$

$$\frac{f(x) - f(1)}{x - 1} = \frac{4x - 4}{x - 1} = 4.$$

So

$$\lim_{x \downarrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \downarrow 1} 4 = 4.$$

Furthermore, for $x < 1$

$$\frac{f(x) - f(1)}{x - 1} = \frac{2x^2 + 2 - 4}{x - 1} = \frac{2(x^2 - 1)}{x - 1} = 2(x + 1).$$

So

$$\lim_{x \uparrow 1} \frac{f(x) - f(1)}{x - 1} = \lim_{x \uparrow 1} 2(x + 1) = 4.$$

Hence, $\lim\limits_{x \to 1} \dfrac{f(x) - f(1)}{x - 1} = 4$. That is: $f$ is differentiable at 1 and $f'(1) = 4$.

Therefore,

$$f'(x) = \begin{cases} 4 & \text{if } x \geq 1 \\ 4x & \text{if } x < 1. \end{cases}$$

8.30 In this case for $x < 1$

$$\frac{f(x) - f(1)}{x - 1} = \frac{2x^2 - 4}{x - 1} = \frac{2(x^2 - 1) - 2}{x - 1} = 2(x + 1) - \frac{2}{x - 1}.$$

In order to show that the limit $\lim\limits_{x \uparrow 1} \left[2(x + 1) - \dfrac{2}{x - 1}\right]$ doesn't exist, we consider the sequence $\big(x_n\big)_{n=1}^{\infty}$ defined by $x_n = 1 - \dfrac{1}{n}$. Obviously, the sequence of images

$$\Big(2 - \frac{2}{n} + 2n\Big)_{n=1}^{\infty}$$

is unbounded, which implies the divergence of the sequence.

Hence, the function $f$ is not differentiable at 1.

**Alternative**

As

$$\lim_{x\downarrow 1} g(x) = 4 \neq 2 = \lim_{x\uparrow 1} g(x),$$

the function $g$ is not continuous at 1. Hence, the function $g$ is not differentiable at 1.

10.5 (a) For $-1 \le x \le 2$,

$$f(x) = \frac{x+5}{\sqrt[3]{x^2}} = x^{-\frac{2}{3}}(x+5) = x^{\frac{1}{3}} + 5x^{-\frac{2}{3}}.$$

Hence, the function $F$ on $[1,2]$, defined by

$$F(x) = \tfrac{3}{4}x^{\frac{4}{3}} + 15x^{\frac{1}{3}} = \tfrac{3}{4}x\sqrt[3]{x} + 15\sqrt[3]{x},$$

is an antiderivative of the function $f$.

(b) For $-1 \le x \le 2$,

$$f(x) = 2\sqrt{x} + \cos x = 2x^{\frac{1}{2}} + \cos x.$$

Hence, the function $F$ on $[1,2]$, defined by

$$F(x) = \tfrac{4}{3}x^{\frac{3}{2}} + \sin x = \tfrac{4}{3}x\sqrt{x} + \sin x,$$

is an antiderivative of the function $f$.

(c) For $-1 \le x \le 2$,

$$f(x) = \frac{1}{(1+x)^2} = (1+x)^{-2}.$$

Hence, the function $F$ on $[1,2]$, defined by

$$F(x) = -(1+x)^{-1} = -\frac{1}{1+x},$$

is an antiderivative of the function $f$.

(d) For $-1 \le x \le 2$,

$$f(x) = \sqrt{2x+1} = (2x+1)^{\frac{1}{2}}.$$

Hence, the function $F$ on $[1,2]$, defined by

$$F(x) = \tfrac{1}{3}(2x+1)^{\frac{3}{2}} = \tfrac{1}{2}\cdot\tfrac{2}{3}(2x+1)\sqrt{2x+1},$$

is an antiderivative of the function $f$.

(e) The function $F$ on $[1,2]$, defined by

$$F(x) = -\cos x^2,$$

is an antiderivative of the function $f$.

(f) For $-1 \le x \le 2$,

$$f(x) = \left(x^2+1\right)^2 = x^4 + 2x^2 + 1.$$

Hence, the function $F$ on $[1,2]$, defined by

$$F(x) = \tfrac{1}{5}x^5 + \tfrac{2}{3}x^3 + x,$$

is an antiderivative of the function $f$.