8.1 (b) Let $c \in \mathbb{R}$. Note that for $x \neq c$,

$$\frac{g(x)-g(c)}{x-c} = \frac{x^3-c^3}{x-c} = \frac{(x-c)(x^2+cx+c^2)}{x-c} = x^2+cx+c^2,$$

which implies that

$$\lim_{x \to c} \frac{g(x)-g(c)}{x-c} = \lim_{x \to c} \left[x^2+cx+c^2\right] = 3c^2.$$

So the function g is differentiable at c. As c was arbitrarily chosen, g is differentiable everywhere and $g'(x) = 3x^2$.

As $g(3) = 27$ and $g'(3) = 27$, the tangent has slope 27 and contains the point $(3, 27)$. An equation of the tangent is $y - 27 = 27(x-3) \Longleftrightarrow y = 27x - 54$.

8.2 Note that for $x \neq 0$

$$\frac{g(x)-g(0)}{x} = \frac{x|x|-0}{x} = |x|,$$

which implies that

$$\lim_{x \to 0} \frac{g(x)-g(0)}{x} = \lim_{x \to 0} |x| = 0.$$

Hence, the function g is differentiable at 0 and $g'(0) = 0$.

8.4 For $x \neq 0$,

$$\frac{f(x)-f(0)}{x} = \begin{cases} \dfrac{x}{x} & \text{if } x > 0 \\ \dfrac{x^2}{x} & \text{if } x < 0 \end{cases} = \begin{cases} 1 & \text{if } x > 0 \\ x & \text{if } x < 0. \end{cases}$$

Apparently,

$$\lim_{x \downarrow 0} \frac{f(x)-f(0)}{x} = 1$$

and

$$\lim_{x \uparrow 0} \frac{f(x)-f(0)}{x} = 0,$$

which implies that the limit $\lim_{x \to 0} \frac{f(x)-f(0)}{x}$ doesn't exist, that is: the function f is not differentiable at 0.

8.5 (a) Let $c > 1$. For $x \neq c$ (and $x > 1$),

$$\frac{f(x)-f(c)}{x-c} = \frac{x^2-2-(c^2-2)}{x-c} = \frac{x^2-c^2}{x-c} = x+c,$$

so that

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = \lim_{x \to c} (x+c) = 2c.$$

Similarly, for $c < 1$,

$$\lim_{x \to c} \frac{f(x)-f(c)}{x-c} = 2c.$$

This proves that the function f is differentiable at c for any $c \neq 1$. For $x \neq 1$, $f'(x) = 2x$.

(b) Obviously,

$$\lim_{x\to 1} f'(x) = \lim_{x\to 1} 2x = 2.$$

(c) The statement is false: see part (d).

(d) Obviously, the function is not continuous at 1:

$$\lim_{x\downarrow 1} f(x) = \lim_{x\downarrow 1}\bigl(x^2-2\bigr) = -1,$$

whereas

$$\lim_{x\uparrow 1} f(x) = \lim_{x\uparrow 1} x = 1.$$

So, according to Theorem 1, the function is not differentiable at 1.

8.6 Let $c \in I$. Then for $x \neq c$,

$$\frac{(5f)(x)-(5f)(c)}{x-c} = \frac{5f(x)-5f(c)}{x-c} = 5\frac{f(x)-f(c)}{x-c}.$$

Hence,

$$\lim_{x\to c}\frac{(5f)(x)-(5f)(c)}{x-c} = \lim_{x\to c} 5\frac{f(x)-f(c)}{x-c} = 5\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = 5f'(c).$$

So the function $5f$ is differentiable at c and $\bigl(5f\bigr)'(c) = 5f'(c)$.

As c was arbitrarily chosen, $5f$ is differentiable and $\bigl(5f\bigr)'(x) = 5f'(x)$.

8.9 Let $n \in \mathbb{N}$. Note that for $x > 0$,

$$g(x) = \frac{1}{f(x)},$$

where $f(x) = x^n$. By Example 10, the function f is differentiable and $f'(x) = nx^{n-1}$. Furthermore, $f(x) \neq 0$ for all $x > 0$.

Then Theorem 4 implies that the function g is differentiable and that for $c > 0$

$$g'(c) = \left(\frac{1}{f}\right)'(c) = -\frac{f'(c)}{f(c)^2} = -\frac{nc^{n-1}}{[c^n]^2} = -\frac{nc^{n-1}}{c^{2n}} = -\frac{n}{c^{n+1}}.$$

8.11 (a) If $g(x) = x + 2x^2$ and $f(t) = 3\sin t$, then

$$h(x) = 3\sin(x+2x^2) = 3\sin g(x) = f\bigl(g(x)\bigr) = (f\circ g)(x).$$

So $h = f\circ g$.

(b) If $g(x) = \sin x$ and $f(t) = 2 + t^2$, then

$$h(x) = 2 + \sin^2 x = 2 + g(x)^2 = f\bigl(g(x)\bigr) = (f\circ g)(x).$$

So $h = f\circ g$.

(c) If $g(x) = 1 - x^2$ and $f(t) = \mathrm{e}^t$, then

$$h(x) = \mathrm{e}^{1-x^2} = \mathrm{e}^{g(x)} = f\bigl(g(x)\bigr) = (f\circ g)(x).$$

So $h = f\circ g$.

(d) If $g(x) = \sin x$ and $f(t) = \sqrt{t}$, then

$$h(x) = \sqrt{\sin x} = \sqrt{g(x)} = f\bigl(g(x)\bigr) = (f\circ g)(x).$$

So $h = f\circ g$.

8.12 (a) For any x,

$$h'(x) = 3\cos(x+2x^2)\cdot(1+4x) = 3(1+4x)\cos(x+2x^2).$$

(b) For any x,

$$h'(x) = 2\sin x \cdot \cos x = 2\sin x \cos x.$$

(c) For any x,

$$h'(x) = \mathrm{e}^{1-x^2}\cdot(-2x) = -2x\,\mathrm{e}^{1-x^2}.$$

(d) For any x satisfying $\sin x > 0$,

$$h'(x) = \frac{1}{2\sqrt{\sin x}}\cdot\cos x = \frac{\cos x}{2\sqrt{\sin x}}.$$

8.27 We know that

$$\lim_{x\to c}\frac{f(x)-f(c)}{x-c} = f'(c) > 0.$$

According to Exercise 5.27, a $\delta > 0$ exist such that

$$\frac{f(x)-f(c)}{x-c} > 0,$$

for all $x \neq c$ in the interval $(c-\delta, c+\delta)$.

Then for any $x \in (c, c+\delta)$, $x - c > 0$, so that $f(x) - f(c) > 0 \Longleftrightarrow f(x) > f(c)$.

8.31 (a) According to the Product Rule and the Chain Rule, for $x \neq 0$

$$f'(x) = \frac{1}{x^2}\cdot 2x\cdot \mathrm{e}^{x^2} + \ln(x^2)\cdot \mathrm{e}^{x^2}\cdot 2x = \mathrm{e}^{x^2}\Big[\frac{2}{x} + 2x\ln(x^2)\Big].$$

(b) According to the Quotient Rule and the Chain Rule, for $x \notin [-1, 1]$,

$$f'(x) = \frac{1}{\frac{x^2-1}{x^2+1}}\cdot\frac{(x^2+1)\cdot 2x - (x^2-1)\cdot 2x}{(x^2+1)^2} = \frac{x^2+1}{x^2-1}\cdot\frac{4x}{(x^2+1)^2} = \frac{4x}{(x^2-1)(x^2+1)} = \frac{4x}{x^4-1}.$$

(c) According to the Quotient Rule and the Chain Rule,

$$f'(x) = \frac{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]\cdot\big[\mathrm{e}^x-\mathrm{e}^{-x}\cdot -1\big] - \big[\mathrm{e}^x-\mathrm{e}^{-x}\big]\cdot\big[\mathrm{e}^x+\mathrm{e}^{-x}\cdot -1\big]}{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]^2} = \frac{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]^2 - \big[\mathrm{e}^x-\mathrm{e}^{-x}\big]^2}{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]^2}$$

$$= \frac{\mathrm{e}^{2x} + 2\,\mathrm{e}^x\mathrm{e}^{-x} + \mathrm{e}^{-2x} - \mathrm{e}^{2x} + 2\,\mathrm{e}^x\mathrm{e}^{-x} - \mathrm{e}^{-2x}}{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]^2} = \frac{4}{\big[\mathrm{e}^x+\mathrm{e}^{-x}\big]^2}.$$

(d) According to the Quotient Rule and the Chain Rule,

$$f'(x) = 2\Big(\frac{\sin x^2}{\cos x^2}\Big)\cdot\frac{\cos x^2\cdot\cos x^2\cdot 2x - \sin x^2\cdot -\sin x^2\cdot 2x}{\cos^2 x^2}$$

$$= \frac{2\sin x^2}{\cos x^2}\cdot\frac{2x\big[\cos^2 x^2 + \sin^2 x^2\big]}{\cos^2 x^2} = \frac{4x\,\sin x^2}{(\cos x^2)^3}.$$