8.15 Note that for $x > 0$

$$f(x) = 3\sqrt[3]{x}\big(\sqrt{x} + x^3\big) = 3\,x^{\frac{1}{3}}\big(x^{\frac{1}{2}} + x^3\big) = 3\,x^{\frac{5}{6}} + 3\,x^{\frac{10}{3}}.$$

According to Exercise 14 and the Arithmetic Rules for differentiable functions, the function f is differentiable and for $x > 0$

$$f'(x) = \tfrac{5}{2}\,x^{-\frac{1}{6}} + 10\,x^{\frac{7}{3}} = \frac{5}{2\sqrt[6]{x}} + 10x^2\sqrt[3]{x}.$$

8.16 We introduce the functions f on $(0,\infty)$, defined by

$$f(x) = \sqrt[3]{x}$$

and the function g on $(-1,\infty)$, defined by

$$g(x) = 1 + x|x|.$$

Then $h(x) = f\big(g(x)\big)$ for all $x > -1$. In other words: the function h is the composition of the functions f and g. Note that f is differentiable on the interval $(0,\infty)$ and that $g\big((-1,\infty)\big) = (0,\infty)$. So if we can prove that the function g is differentiable on the interval $(-1,\infty)$, then the function h is differentiable on the interval $(-1,\infty)$.

In order to prove that the function g is differentiable we note that

$$g(x) = \begin{cases} 1 - x^2 & \text{if } -1 < x < 0 \\ 1 + x^2 & \text{if } x \geq 0. \end{cases}$$

In view of the Arithmetic Rules for differentiable functions, the function g is differentiable for $x > -1$ and $x \neq 0$. For $x \neq 0$

$$\frac{g(x) - g(0)}{x} = \frac{g(x) - 1}{x} = \frac{1 + x|x| - 1}{x} = |x|.$$

Hence,

$$\lim_{x \to 0} \frac{g(x) - g(0)}{x} = \lim_{x \to 0} |x| = 0.$$

This means that the function g is also differentiable at 0 and that $g'(0) = 0$.

So the function g is differentiable and

$$g'(x) = \begin{cases} -2x & \text{if } -1 < x < 0 \\ 0 & \text{if } x = 0 \\ 2x & \text{if } x > 0 \end{cases} = 2|x|.$$

According to the Chain Rule,

$$h'(x) = f'\big(g(x)\big) \cdot g'(x) = \tfrac{2}{3}(1 + x|x|)^{-\frac{2}{3}}\,|x|.$$

8.18 The function cos restricted to the interval $[0, \pi]$ is strictly decreasing and therefore invertible. According to the Inverse Function Theorem, the function arccos is differentiable on $(-1, 1)$ and for $y \in (-1, 1)$,

$$\arccos'(y) = \frac{1}{\cos'\big(\arccos(y)\big)} = \frac{1}{-\sin\big(\arccos(y)\big)} = -\frac{1}{\sqrt{1-\cos^2\big(\arccos(y)\big)}} = -\frac{1}{\sqrt{1-y^2}}.$$

8.19 Since, for $x > 0$,

$$h(x) = 4x^{\frac{3}{2}} = 4x\sqrt{x},$$

the function h is the product of the functions f and g on $(0, \infty)$ defined by $f(x) = 4x$ and $g(x) = \sqrt{x}$, respectively. Since the functions f and g are differentiable, the Product Rule implies that the function h is differentiable. Furthermore, for $x > 0$,

$$h'(x) = f'(x) \cdot g(x) + f(x) \cdot g'(x) = 4\sqrt{x} + 4x \cdot \frac{1}{2\sqrt{x}} = 6\sqrt{x}.$$

This implies that for all $x > 0$,

$$E_h(x) = h'(x) \cdot \frac{x}{h(x)} = 6\sqrt{x} \cdot \frac{x}{4x\sqrt{x}} = \tfrac{3}{2}.$$

8.21 (a) Since, for $p > 0$, $r(p) = p \cdot d(p)$,

$$r'(p) = d(p) + pd'(p) = d(p) + d'(p)\frac{p}{d(p)} \cdot d(p) = d(p)\left[1 + E_d(p)\right].$$

(b) By using part (a), it follows that

$$E_r(p) = r'(p) \cdot \frac{p}{r(p)} = d(p)\left[1 + E_d(p)\right] \cdot \frac{p}{p \cdot d(p)} = 1 + E_d(p).$$

8.22 (a) For $-1 < u < 1$,

$$\begin{aligned} h'(u) &= (10A)^{2u}\ln(10A) \cdot 2 \cdot \sqrt[3]{1-u^2} + (10A)^{2u}\frac{1}{3\sqrt[3]{(1-u^2)^2}} \cdot (-2u) \\ &= (10A)^{2u}\frac{6\ln(10A)(1-u^2) - 2u}{3\sqrt[3]{(1-u^2)^2}}. \end{aligned}$$

(b) For all $k \neq 2$,

$$\begin{aligned} G'(k) &= \frac{(2k-4)\dfrac{1}{2\sqrt{(bk)^2+3c^2}} \cdot 2b^2k - \sqrt{(bk)^2+3c^2} \cdot 2}{(2k-4)^2} \\ &= \frac{(2k-4)b^2k - 2\left[(bk)^2+3c^2\right]}{(2k-4)^2\sqrt{(bk)^2+3c^2}} \\ &= \frac{2b^2k^2 - 4b^2k - 2(bk)^2 - 6c^2}{(2k-4)^2\sqrt{(bk)^2+3c^2}} = \frac{-4b^2k - 6c^2}{(2k-4)^2\sqrt{(bk)^2+3c^2}}. \end{aligned}$$

(c) For all u,

$$f'(u) = \frac{1}{1+\dfrac{1}{(1+u^2)^2}} \cdot \frac{-2u}{(1+u^2)^2} = \frac{-2u}{(1+u^2)^2+1}.$$

(d)

$$C'(h) = \frac{1}{1+\tan^2 h} \cdot 2\tan h \cdot \frac{1}{\cos^2 h} = \frac{2\tan h}{\cos^2 h + \sin^2 h} = 2\tan h.$$

8.23 As $f(1) = 5 = f(-1)$, the function f is not invertible!

10.1 (a) Take $a = 0, b = 1, f(t) = \dfrac{1}{\sqrt{t}}$ and $\varphi(x) = 3x^2 + 1$.

(b) An antiderivative of f is the function F defined by $F(t) = 2\sqrt{t}$.

(c) As $\varphi'(x) = 2$, we obtain

$$\int_0^1 \frac{x}{\sqrt{3x^2+1}}\,\mathrm{d}x = \tfrac{1}{6}\int_0^1 \frac{1}{\sqrt{3x^2+1}} \cdot 6x\,\mathrm{d}x = \tfrac{1}{6}\int_0^1 \frac{1}{\sqrt{\varphi(x)}} \cdot \varphi'(x)\,\mathrm{d}x$$
$$= \tfrac{1}{3}\Big[\sqrt{\varphi(x)}\Big]_0^1 = \tfrac{1}{3}\Big[\sqrt{3x^2+1}\Big]_0^1 = \tfrac{1}{3}.$$

11.2 (a) Take $a = 0, b = 2, f(t) = \dfrac{1}{t}$ and $\varphi(x) = 2x + 1$.

(b) An antiderivative of f is the function F defined by $F(t) = \ln t$.

(c) As $\varphi'(x) = 2$, we obtain

$$\int_0^2 \frac{1}{2x+1}\,\mathrm{d}x = \tfrac{1}{2}\int_0^2 \frac{1}{2x+1} \cdot 2\,\mathrm{d}x = \tfrac{1}{2}\int_0^2 \frac{1}{\varphi(x)} \cdot \varphi'(x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\Big[\ln\varphi(x)\Big]_0^2 = \tfrac{1}{2}\Big[\ln(2x+1)\Big]_0^2 = \tfrac{1}{2}\ln 5.$$

11.4 (a) In order to evaluate the integral

$$\int_{-1}^1 \frac{x^2}{\sqrt{x^3+9}}\,\mathrm{d}x$$

we introduce $\varphi(x) = x^3 + 9$. As $\varphi(-1) = 8$ and $\varphi(1) = 10$, the Method of Substitution leads to

$$\int_{-1}^1 \frac{x^2}{\sqrt{x^3+9}}\,\mathrm{d}x = \int_{-1}^1 \frac{1}{3\sqrt{x^3+9}} \cdot 3x^2\,\mathrm{d}x = \int_{-1}^1 \frac{1}{3\sqrt{\varphi(x)}} \cdot \varphi'(x)\,\mathrm{d}x$$
$$= \int_8^{10} \tfrac{1}{3}t^{-\frac{1}{2}}\,\mathrm{d}t = \Big[\tfrac{2}{3}t^{\frac{1}{2}}\Big]_8^{10} = \tfrac{2}{3}\sqrt{10} - \tfrac{2}{3}\sqrt{8}.$$

(b) In order to evaluate the integral

$$\int_0^{\ln 2} \mathrm{e}^x\big(1+\mathrm{e}^x\big)\,\mathrm{d}x$$

we introduce $\varphi(x) = 1 + \mathrm{e}^x$. As $\varphi(0) = 2$ and $\varphi(\ln 2) = 3$, the Method of Substitution leads to

$$\int_0^{\ln 2} \mathrm{e}^x\big(1+\mathrm{e}^x\big)\,\mathrm{d}x = \int_0^{\ln 2} \big(1+\mathrm{e}^x\big) \cdot \mathrm{e}^x\,\mathrm{d}x = \int_0^{\ln 2} \varphi(x) \cdot \varphi'(x)\,\mathrm{d}x$$
$$= \int_2^3 t\,\mathrm{d}t = \Big[\tfrac{1}{2}t^2\Big]_2^3 = 4\tfrac{1}{2} - 2 = 2\tfrac{1}{2}.$$