8.14 We introduce the functions f and g on $(0,\infty)$, defined by

$$f(x) = x^m$$

and

$$g(x) = x^{\frac{1}{n}}.$$

Then, for $x > 0$,

$$h(x) = x^{\frac{m}{n}} = \left[x^{\frac{1}{n}}\right]^m = f\big(g(x)\big).$$

In other words: the function h is the composition of the differentiable functions f and g. So according to the Chain Rule, the function h is differentiable and for $x > 0$

$$\begin{aligned} h'(x) &= f'\big(g(x)\big) \cdot g'(x) = m\, g(x)^{m-1} \cdot \frac{1}{n} x^{\frac{1}{n}-1} \\ &= m \left[x^{\frac{1}{n}}\right]^{m-1} \cdot \frac{1}{n} x^{\frac{1}{n}-1} = \frac{m}{n} x^{\frac{m-1}{n}} \cdot x^{\frac{1}{n}-1} = \frac{m}{n} x^{\frac{m}{n}-1}. \end{aligned}$$

8.17 (a) For $-\frac{1}{2}\pi < t < \frac{1}{2}\pi$,

$$\frac{1}{\cos^2 t} = \frac{\sin^2 t + \cos^2 t}{\cos^2 t} = 1 + \frac{\sin^2 t}{\cos^2 t} = 1 + \tan^2 t.$$

(b) For $y \in \mathbb{R}$

$$\arctan'(y) = \frac{1}{\tan'(\arctan(y))} = \frac{1}{\dfrac{1}{\cos^2(\arctan(y))}} = \frac{1}{1 + \tan^2[\arctan(y)]} = \frac{1}{1+y^2}.$$

8.20 According to the Product Rule, for a $c \in I$,

$$\begin{aligned} E_{f\cdot g}(c) &= \big(f \cdot g\big)'(c) \cdot \frac{c}{\big(f\cdot g\big)(c)} = [f'(c)g(c) + f(c)g'(c)] \cdot \frac{c}{f(c)g(c)} \\ &= f'(c)g(c) \cdot \frac{c}{f(c)g(c)} + f(c)g'(c) \cdot \frac{c}{f(c)g(c)} \\ &= f'(c) \cdot \frac{c}{f(c)} + g'(c) \cdot \frac{c}{g(c)} = E_f(c) + E_g(c). \end{aligned}$$

8.24 (a) Say $\lim\limits_{x\to 0} \dfrac{f(x)}{x} = \ell$. For every $x \neq 0$

$$f(x) = x \cdot \frac{f(x)}{x}.$$

According to the Product Rule for limits of functions, this implies that

$$f(0) = \lim_{x\to 0} f(x) = \lim_{x\to 0} x \cdot \frac{f(x)}{x} = 0 \cdot \ell = 0.$$

Here the first equality is a consequence of the continuity of the function f at 0.

(b) The function is differentiable at 0, because

$$\lim_{x\to 0} \frac{f(x) - f(0)}{x - 0} = \lim_{x\to 0} \frac{f(x)}{x} = \ell.$$

Apparently, $f'(0) = \ell$.

8.26 Note that for all $x \neq 1$,

$$\frac{f(x)-f(1)}{x-1} = \frac{3x+2|x-1|^{\frac{3}{2}}-3}{x-1} = \frac{3(x-1)+2|x-1|^{\frac{3}{2}}}{x-1} = \begin{cases} 3+2\sqrt{x-1} & \text{if } x>1 \\ 3-2\sqrt{1-x} & \text{if } x<1. \end{cases}$$

Let $\big(x_n\big)_{n=1}^{\infty}$ be a sequence converging to 1 such that $x_n > 1$ for all n. Then

$$\lim_{n\to\infty} \frac{f(x_n)-f(1)}{x_n-1} = \lim_{n\to\infty} \big[3+2\sqrt{x_n-1}\big] = 3.$$

As the sequence $\big(x_n\big)_{n=1}^{\infty}$ was chosen arbitrarily, this proves that

$$\lim_{x\downarrow 1} \frac{f(x)-f(1)}{x-1} = 3.$$

In a similar way one proves that the left-hand derivative of f at 1 is 3. Hence, the function f is differentiable at 1 and $f'(1) = 3$.

Alternative

It is also possible to apply the Arithmetic Rules for limits of functions to show that the two limits discussed before are equal to 3.

11.3 (a) Take $a = 0, b = 4, f(t) = \sqrt{t}$ and $\varphi(x) = x^2 + 9$.

(b) An antiderivative of f is the function F defined by $F(t) = \frac{2}{3}t\sqrt{t}$.

(c) As $\varphi'(x) = 2x$, we obtain

$$\int_0^4 x\sqrt{x^2+9}\,\mathrm{d}x = \tfrac{1}{2}\int_0^4 \sqrt{x^2+9}\cdot 2x\,\mathrm{d}x = \tfrac{1}{2}\int_0^4 \sqrt{\varphi(x)}\cdot\varphi'(x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\Big[\tfrac{2}{3}\varphi(x)\sqrt{\varphi(x)}\Big]_0^4 = \Big[\tfrac{1}{3}(x^2+9)\sqrt{x^2+9}\Big]_0^4 = \tfrac{1}{3}(125-27) = \tfrac{98}{3}.$$

11.4 (d) In order to evaluate the integral

$$\int_0^{\frac{1}{6}\pi} \tan x\,\mathrm{d}x = \int_0^{\frac{1}{6}\pi} \frac{\sin x}{\cos x}\,\mathrm{d}x$$

we introduce $\varphi(x) = \cos x$. As $\varphi(0) = 1$ and $\varphi(\frac{1}{6}\pi) = \frac{1}{2}\sqrt{3}$, the Method of Substitution leads to

$$\int_0^{\frac{1}{6}\pi} \frac{\sin x}{\cos x}\,\mathrm{d}x = -\int_0^{\frac{1}{6}\pi} \frac{1}{\cos x}\cdot -\sin x\,\mathrm{d}x = -\int_0^{\frac{1}{6}\pi} \frac{1}{\varphi(x)}\cdot\varphi'(x)\,\mathrm{d}x$$
$$= -\int_1^{\frac{1}{2}\sqrt{3}} \frac{1}{t}\,\mathrm{d}t = -\Big[\ln t\Big]_1^{\frac{1}{2}\sqrt{3}} = -\ln\tfrac{1}{2}\sqrt{3}.$$

11.5 Note that

$$\frac{1}{x^2+x+1} = \frac{1}{x^2+x+\frac{1}{4}+\frac{3}{4}} = \frac{1}{(x+\frac{1}{2})^2+\frac{3}{4}} = \frac{\frac{4}{3}}{\frac{4}{3}(x+\frac{1}{2})^2+1} = \frac{4}{3}\frac{1}{[\frac{2}{3}\sqrt{3}(x+\frac{1}{2})]^2+1}.$$

We use the Method of Substitution with $\varphi(x) = \frac{2}{3}\sqrt{3}(x+\frac{1}{2})$. Then we obtain

$$\int_1^4 \frac{1}{x^2+x+1}\,\mathrm{d}x = \int_1^4 \frac{\frac{4}{3}}{\frac{2}{3}\sqrt{3}}\,\frac{1}{\varphi(x)^2+1}\,\varphi'(x)\,\mathrm{d}x = \tfrac{2}{3}\sqrt{3}\,[\arctan\varphi(x)]_1^4$$
$$= \tfrac{2}{3}\sqrt{3}\big(\arctan 3\sqrt{3} - \arctan\sqrt{3}\big).$$