9.1 Since the function f is differentiable everywhere, Theorem 1 implies that if f has an extreme value at c, then $f'(c) = 0$. Since $f'(c) = 0 \iff c = 0$, the proof is complete if we can show that the function doesn't have an extreme value at 0.

Let $\varepsilon > 0$. Then $\pm\frac{1}{2}\varepsilon \in (-\varepsilon, \varepsilon)$ and

$$f(\tfrac{1}{2}\varepsilon) = \tfrac{1}{8}\varepsilon^3 > 0 = f(0)$$

and

$$f(-\tfrac{1}{2}\varepsilon) = -\tfrac{1}{8}\varepsilon^3 < 0 = f(0).$$

This proves that any interval around 0 contains a point where the function is larger than $f(0)$: so the function doesn't take on a maximum at 0. Also any interval around 0 contains a point where the function is smaller than $f(0)$: so the function doesn't take on a minimum at 0.

9.2 (a) Using

$$f'(x) = 3x^2 - 6x - 9 = 3(x^2 - 2x - 3) = 3(x-3)(x+1),$$

we get

$$f'(x) = 0 \iff 3(x-3)(x+1) = 0 \iff x = -1 \text{ or } x = 3.$$

So the stationary points of the the function f are -1 and 3.

(b) Using

$$g'(u) = 3Au^2 + 2Bu + C,$$

we get

$$g'(u) = 0 \iff 3Au^2 + 2Bu + C = 0 \iff u = \frac{-2B \pm \sqrt{4B^2 - 12AC}}{6A},$$

provided that $A \neq 0$ and $B^2 - 3AC \geq 0$.

So the stationary points of the function g are $\dfrac{-B + \sqrt{B^2 - 3AC}}{3A}$ and $\dfrac{-B - \sqrt{B^2 - 3AC}}{3A}$.

(c) Using

$$F'(v) = \frac{1}{1 + (v-2)^2} \cdot 2(v-2),$$

we get

$$F'(v) = 0 \iff \frac{2(v-2)}{1 + (v-2)^2} = 0 \iff v = 2.$$

So the stationary point of the the function F is 2.

(d) Using

$$h'(y) = -\mathrm{e}^{-y}(y^2 - 2y - 7) + \mathrm{e}^{-y}(2y - 2) = -\mathrm{e}^{-y}(y^2 - 4y - 5) = -\mathrm{e}^{-y}(y-5)(y+1),$$

we get

$$h'(y) = 0 \iff (y-5)(y+1) = 0 \iff y = -1 \text{ or } y = 5.$$

So the stationary points of the the function h are -1 and 5.

9.3 Assume that there exist points x and x' in $[a,b]$ such that $x \neq x'$ and $g(x) = g(x')$. Say $x < x'$.

The the function g restricted to the interval $[x, x']$ is continuous and differentiable on the interval (x, x'). Furthermore, $g(x) = g(x')$.

Then, according to Rolle's Theorem, there exists a $\tau \in (x, x')$ such that $g'(\tau) = 0$. This is in contradiction with the data of the exercise.

9.4 We introduce the function f on $\mathbb{R}$, defined by $f(x) = x^5 + 2x^3 + x - 5$. Obviously, a number z is a solution of the equation if and only if z is a zero of the function f. We will prove that the function f has a unique zero.

(a) Note that $f(0) = -5 < 0$ and $f(2) = 45 > 0$.

Since the function f is the sum of continuous functions, according to the arithmetic rules for continuous functions, the function f is continuous on the interval $[0, 2]$.

According to the Intermediate Value Theorem there exists a $\tau \in (0, 2)$ such that $f(\tau) = 0$. So the function f has at least one zero.

(b) Note that the function f is differentiable and that for all $x \in \mathbb{R}$

$$f'(x) = 5x^4 + 6x^2 + 1 > 0.$$

Assume that c and d are zeros of the function f and that $c < d$.

The function f restricted to the interval $[c, d]$ is continuous and differentiable on the interval (c, d); furthermore $f(c) = f(d) = 0$.

According to Rolle's Theorem there exists a $\tau \in (c, d)$ such that $f'(\tau) = 0$. This is in contradiction with the fact that $f'(x) > 0$ for all $x \in \mathbb{R}$.

9.5 The function f restricted to the interval $[a, b]$ is continuous and differentiable on the interval (a, b). According to the Mean Value Theorem a number $\tau \in (a, b)$ exists such that

$$f'(\tau) = \frac{f(b) - f(a)}{b - a} \Longleftrightarrow 2\tau = \frac{b^2 - a^2}{b - a} \Longleftrightarrow 2\tau = b + a \Longleftrightarrow \tau = \tfrac{1}{2}(a + b).$$

9.6 We introduce the function $h = f - g$. Then the function h is continuous on the interval $[a, b]$ and differentiable on the interval (a, b). Furthermore, for every $x \in (a, b)$,

$$h'(x) = f'(x) - g'(x) = 0.$$

In view of Theorem 4 this means that a constant C exists such that $h = C$, or: $f = g + C$.

9.9 (a) According to Theorem 5(a), the function f is increasing. Assume that the function f is not **strictly** increasing. Then there exits two points in I, say x and x', such that $x < x'$ and $f(x) = f(x')$.

Since f is increasing on the interval $[x, x']$, it follows that f is constant on that interval.

Hence, f' is zero throughout the open interval (x, x').

(b) Note that

$$g'(x) = 1 + \cos x.$$

So $g'(x) \geq 0$ for all x and $g'(x) > 0$ except if $\cos x$ is equal to -1. Since $\cos x = -1$ if and only if $|x|$ is an odd multiple of π, part (a) implies that g is strictly increasing.

9.10 Note that for $x \neq 0$,

$$f'(x) = 1 - \frac{1}{x^2} = \frac{x^2-1}{x^2} = \frac{(x-1)(x+1)}{x^2}.$$

Hence, $f'(x) = 0 \iff x = -1$ or $x = 1$ and the sign survey of f' is given by

	-1		0		1		
$+\ +$	0	$-\ -$	$\times$	$-\ -$	0	$+\ +$	f'

According to Theorem 5, the function f is strictly increasing on the intervals $[1, \infty)$ and $(-\infty, -1]$. The function is strictly decreasing on the intervals $[-1, 0)$ and $(0, 1]$.

9.13 If the function f has a local minimum at c, then Theorem 1 implies that $f'(c) = 0$.
Now assume that $f''(c) < 0$. Then Theorem 7 implies that f has a local maximum at c.
This is impossible unless the function f is constant in the neighborhood of c. Then however $f''(c) = 0$ which is not in accordance with our assumption. So $f''(c) \leq 0$.

9.14 Note that the function f is twice differentiable on the set $\mathbb{R} \setminus \{0\}$ and that for $x \neq 0$,

$$f'(x) = 1 - \frac{1}{x^2}$$

and

$$f''(x) = \frac{2}{x^3}.$$

So $f'(x) = 0 \iff x^2 = 1 \iff x = \pm 1$.
Since $f''(-1) = -2 < 0$ and -1 is an interior point of the interval $(-\infty, 0)$, the function has a maximum at -1.
Because $f''(1) = 2 > 0$ and 1 is an interior point of the interval $(0, \infty)$, the function has a minimum at 1.

11.6 (a) In order to evaluate the integral

$$\int_0^1 x^2\, \mathrm{e}^x \, \mathrm{d}x$$

we introduce $f(x) = x^2$ and $g'(x) = \mathrm{e}^x$. Because, for any $x > 0$, $f'(x) = 2x$ and $g(x) = \mathrm{e}^x$, Partial Integration leads to

$$\int_0^1 x^2\, \mathrm{e}^x \, \mathrm{d}x = \left[x^2 \cdot \mathrm{e}^x\right]_0^1 - \int_0^1 2x \cdot \mathrm{e}^x \, \mathrm{d}x = \mathrm{e} - 2.$$

(d) In order to evaluate the integral

$$\int_1^{\mathrm{e}} (\ln x)^2 \cdot 1 \, \mathrm{d}x$$

we introduce $f(x) = (\ln x)^2$ and $g'(x) = 1$. Because, for any $x > 0$, $f'(x) = \dfrac{2 \ln x}{x}$ and $g(x) = x$, Partial Integration leads to

$$\int_1^{\mathrm{e}} (\ln x)^2 \, \mathrm{d}x = \left[(\ln x)^2 \cdot x\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \frac{2 \ln x}{x} \cdot x \, \mathrm{d}x = \mathrm{e} - 2 \int_1^{\mathrm{e}} \ln x \, \mathrm{d}x = \mathrm{e} - 2.$$