9.7 (a) Let $c > 0$. According to the Arithmetic Rules for continuous or differentiable functions, the function $h = g - f$ restricted to the interval $[0, c]$ is continuous and differentiable on the interval $(0, c)$. By the Mean Value Theorem there exists a number $\tau \in (0, c)$ such that

$$\frac{h(c) - h(0)}{c} = h'(\tau) \Longrightarrow \frac{g(c) - f(c) - [g(0) - f(0)]}{c} = g'(\tau) - f'(\tau)$$
$$\Longrightarrow g(c) - f(c) = c[g'(\tau) - f'(\tau)] \geq 0.$$

So $g(c) \geq f(c)$. As c was arbitrarily chosen, this implies that $g \geq f$.

(b) Let f and g be the functions defined by $f(x) = \sin x$ and $g(x) = x$. Then $f(0) = 0 = g(0)$ and $f'(x) = \cos x \leq 1 = g'(x)$ for all $x > 0$.

So, according to part (a), for all $x \geq 0$,

$$f(x) \leq g(x) \Longrightarrow \sin x \leq x.$$

(c) Let f and g be the functions defined by $f(x) = -\cos x$ and $g(x) = -1 + \frac{1}{2}x^2$. Then $f(0) = -1 = g(0)$ and $f'(x) = \sin x \leq x = g'(x)$ for all $x > 0$.

So, according to part (a), for all $x \geq 0$,

$$f(x) \leq g(x) \Longrightarrow -\cos x \leq -1 + \tfrac{1}{2}x^2 \Longrightarrow \cos x \geq 1 - \tfrac{1}{2}x^2.$$

Hence, for all $x > 0$,

$$1 - \tfrac{1}{2}x^2 \leq \cos x \leq 1 \Longrightarrow -\tfrac{1}{2}x \leq \frac{\cos x - 1}{x} \leq 0.$$

Since $\lim_{x \downarrow 0} -\frac{1}{2}x = 0$, the Sandwich Lemma for functions implies that

$$\lim_{x \downarrow 0} \frac{\cos x - 1}{x} = 0.$$

9.8 Let $x < c$. Because the restriction of the function f to the interval $[x, c]$ is continuous and differentiable on the interval (x, c), the Mean Value Theorem implies the existence of a $\tau \in (x, c)$ such that

$$\frac{f(c) - f(x)}{c - x} = f'(\tau) > 0 \Longrightarrow f(c) - f(x) > 0 \Longrightarrow f(x) < f(c) = 0.$$

The case $x > c$ can be handled in a similar way.

9.11 Because $f'(x) < 0$ for all $x \in (a, c)$, f is strictly decreasing on the interval $(a, c]$. So, for every $x \in (a, c)$, $f(x) > f(c)$.

Similarly, $f(x) > f(c)$ for all $x \in (c, b)$.

Hence, $f(c)$ is the smallest value of the function on the interval (a, b).

9.12 According to the Quotient Rule for differentiable functions, the function g is differentiable on the interval $(0, 1)$ and for $x \in (0, 1)$,

$$g'(x) = \frac{xf'(x) - f(x)}{x^2} = \frac{1}{x}\Big(f'(x) - \frac{f(x)}{x}\Big).$$

Let $c \in (0,1)$. Because the function f restricted to the interval $[0,c]$ is continuous and differentiable on the interval $(0,c)$, according to the Mean Value Theorem, there exist a $\tau \in (0,c)$ such that

$$f'(\tau) = \frac{f(c) - f(0)}{c - 0} = \frac{f(c)}{c}.$$

This however means that

$$g'(c) = \frac{1}{c}\Big(f'(c) - \frac{f(c)}{c}\Big) = \frac{1}{c}\big(f'(c) - f'(\tau)\big) \geq 0.$$

Note that $f'(c) \geq f'(\tau)$, because the function f' is increasing and because $\tau < c$.
Since c was arbitrarily chosen, $g' \geq 0$. So the function g is increasing on the interval $(0,1)$.

11.6 (b) In order to evaluate the integral

$$\int_0^\pi x^2 \sin x \, \mathrm{d}x$$

we introduce $f(x) = x^2$ and $g'(x) = \sin x$. Because, for any $x > 0$, $f'(x) = 2x$ and $g(x) = -\cos x$, Partial Integration leads to

$$\int_0^\pi x^2 \sin x \, \mathrm{d}x = \big[x^2 \cdot -\cos x\big]_0^\pi - \int_0^\pi 2x \cdot -\cos x \, \mathrm{d}x = \pi^2 + 2\int_0^\pi x \cos x \, \mathrm{d}x$$

In order to evaluate the integral

$$\int_0^\pi x \cos x \, \mathrm{d}x$$

we introduce $f(x) = x$ and $g'(x) = \cos x$. Because, for any $x > 0$, $f'(x) = 1$ and $g(x) = \sin x$, Partial Integration leads to

$$\int_0^\pi x \cos x \, \mathrm{d}x = \big[x \cdot \sin x\big]_0^\pi - \int_0^\pi 1 \cdot \sin x \, \mathrm{d}x = \big[\cos x\big]_0^\pi = -1 - 1 = -2.$$

So the given integral is equal to $\pi^2 - 4$.

(c) In order to evaluate the integral

$$\int_0^\pi \sin^2 x \, \mathrm{d}x$$

we introduce $f(x) = \sin x$ and $g'(x) = \sin x$. Because, for any $x > 0$, $f'(x) = \cos x$ and $g(x) = -\cos x$, Partial Integration leads to

$$\begin{aligned}\int_0^\pi \sin^2 x \, \mathrm{d}x &= \big[\sin x \cdot -\cos x\big]_0^\pi - \int_0^\pi \cos x \cdot -\cos x \, \mathrm{d}x = \int_0^\pi \cos^2 x \, \mathrm{d}x = \int_0^\pi \big[1 - \sin^2 x\big] \, \mathrm{d}x \\ &= \int_0^\pi \mathrm{d}x - \int_0^\pi \sin^2 x \, \mathrm{d}x = \pi - \int_0^\pi \sin^2 x \, \mathrm{d}x.\end{aligned}$$

Hence,

$$\int_0^\pi \sin^2 x \, \mathrm{d}x = \tfrac{1}{2}\pi.$$

11.12 In order to evaluate the integral

$$\int_0^1 \frac{x^2}{\sqrt{4-3x}}\,\mathrm{d}x$$

we use the method of Partial Integration with $f(x) = x^2$ and $g'(x) = \dfrac{1}{\sqrt{4-3x}}$. Then

$$\int_0^1 \frac{x^2}{\sqrt{4-3x}}\,\mathrm{d}x = \left[-\tfrac{2}{3}\sqrt{4-3x}\cdot x^2\right]_0^1 - \int_0^1 -\tfrac{4}{3}x\sqrt{4-3x}\,\mathrm{d}x = -\tfrac{2}{3} + \tfrac{4}{3}\int_0^1 x\sqrt{4-3x}\,\mathrm{d}x.$$

This last integral can be evaluated by using the method of Partial Integration with $f(x) = x$ and $g'(x) = \sqrt{4-3x}$. Then we obtain

$$\int_0^1 x\sqrt{4-3x}\,\mathrm{d}x = \left[-\tfrac{2}{9}(4-3x)^{1\frac{1}{2}}\cdot x\right]_0^1 - \int_0^1 -\tfrac{2}{9}(4-3x)^{1\frac{1}{2}}\,\mathrm{d}x$$

$$= -\tfrac{2}{9} + \tfrac{2}{9}\left[-\tfrac{2}{15}(4-3x)^{2\frac{1}{2}}\right]_0^1 = \tfrac{2}{9}\left(-1 - \tfrac{2}{15} + \tfrac{64}{15}\right) = \tfrac{2}{9}\cdot\tfrac{47}{15}.$$

The original integral is equal to

$$-\tfrac{2}{3} + \tfrac{4}{3}\cdot\tfrac{2}{9}\cdot\tfrac{47}{15} = \tfrac{106}{405}.$$

11.13 In order to evaluate the integral

$$\int_1^{\mathrm{e}} \sqrt{x}\ln x\,\mathrm{d}x$$

we use the method of Partial Integration with $f(x) = \ln x$ and $g'(x) = \sqrt{x}$. Then

$$\int_1^{\mathrm{e}} \sqrt{x}\ln x\,\mathrm{d}x = \left[\ln x\cdot\tfrac{2}{3}x^{1\frac{1}{2}}\right]_1^{\mathrm{e}} - \int_1^{\mathrm{e}} \tfrac{2}{3}x^{1\frac{1}{2}}\cdot\frac{1}{x}\,\mathrm{d}x = \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \int_1^{\mathrm{e}} \tfrac{2}{3}x^{\frac{1}{2}}\,\mathrm{d}x$$

$$= \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \left[\tfrac{4}{9}x^{1\frac{1}{2}}\right]_1^{\mathrm{e}} = \tfrac{2}{3}\mathrm{e}\sqrt{\mathrm{e}} - \tfrac{4}{9}\mathrm{e}\sqrt{\mathrm{e}} + \tfrac{4}{9} = \tfrac{2}{9}\mathrm{e}\sqrt{\mathrm{e}} + \tfrac{4}{9}.$$