9.17 Since $f(1) = 2$ and $f'(1) = 0$, the linear approximation ℓ_1 of f at 1 is given by

$$\ell_1(x) = f(1) + f'(1)(x-1) = 2.$$

Further, $f''(x) = 2x^{-3}$. So for $x \in (0,2)$ and $x \neq 1$ the remainder is

$$r(x) = \frac{f''(\tau)}{2}(x-1)^2 = \frac{1}{\tau^3}(x-1)^2,$$

where τ is between 1 and x.

9.20 (a) As $f'(x) = 1 - x^{-2}$, we get $f''(x) = 2x^{-3}$ and $f^{(3)}(x) = -6x^{-4}$. Hence, $f''(1) = 2$ and $f^{(3)}(1) = -6$ and the Taylor polynomial p_3 of degree 3 for f at 1 is given by

$$p_3(x) = 2 + (x-1)^2 - (x-1)^3.$$

9.23 (a) We introduce the functions f and g on the interval $(0, \infty)$, defined by $f(x) = \ln x$ and $g(x) = x - 1$. Note that $x \mapsto \ln x$ and $x \mapsto x - 1$ are continuous functions on the interval $(0, \infty)$. Moreover, $f(1) = g(1) = 0$.

The functions f and g are differentiable and $f'(x) = \dfrac{1}{x}$ and $g'(x) = 1$, so that

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{1}{x} = 1.$$

According to the weak form of de l'Hôpital's rule, $\displaystyle\lim_{x \to 1} \frac{f(x)}{g(x)} = 1$.

(b) We introduce the functions f and g on the interval $(0, \infty)$, defined by $f(x) = |\ln x|$ and $g(x) = |x^2 - 1|$. Note that $x \mapsto \ln x$ and $x \mapsto x^2 - 1$ are continuous functions on the interval $(0, \infty)$. Then, according to Exercise 5.12, f and g are continuous functions on the interval $(0, \infty)$.

Moreover, $f(1) = g(1) = 0$.

The function f is differentiable for $x \neq 1$ and

$$f'(x) = \begin{cases} \dfrac{1}{x} & \text{if } x > 1 \\ -\dfrac{1}{x} & \text{if } 0 < x < 1. \end{cases}$$

The function g is differentiable for $x \neq 1$ and

$$g'(x) = \begin{cases} 2x & \text{if } x > 1 \\ -2x & \text{if } 0 < x < 1. \end{cases}$$

Since for $x \neq 1$

$$\frac{f'(x)}{g'(x)} = \frac{1}{2x^2},$$

the following holds:

$$\lim_{x \to 1} \frac{f'(x)}{g'(x)} = \lim_{x \to 1} \frac{1}{2x^2} = \tfrac{1}{2}.$$

According to the strong form of de l'Hôpital's rule, $\displaystyle\lim_{x \to 1} \frac{f(x)}{g(x)} = \tfrac{1}{2}$.

Alternative Instead you could evaluate the limit $\lim_{x \to 1} \frac{\ln x}{x^2 - 1}$ first. As before you can prove that this limit equals $\frac{1}{2}$. Then use the result that

$$\lim_{x \to 1} \frac{|\ln x|}{|x^2 - 1|} = \lim_{x \to 1} \left| \frac{\ln x}{x^2 - 1} \right| = \left|\tfrac{1}{2}\right| = \tfrac{1}{2}.$$

9.27 Note that $p(0) = b < 0$ and

$$p\big(-\frac{b}{a}\big) = -\frac{b^3}{a^3} - \frac{b}{a} \cdot a + b = -\frac{b^3}{a^3} > 0.$$

The function p restricted to the interval $\big[0, -\frac{b}{a}\big]$ is continuous. Hence, the Intermediate Value Theorem implies that a $\tau \in \big(0, -\frac{b}{a}\big)$ exists such that $p(\tau) = 0$.
Finally, as $p'(x) = 3x^2 + a > 0$ for all x, the function p is strictly increasing. Hence, p has a unique zero.

9.33 Let $x > 0$. The function f restricted to the interval $[x, x+2]$ is continuous and differentiable on the interval $(x, x+2)$. According to the Mean Value Theorem, there exists a $\tau \in (x, x+2)$ such that

$$f'(\tau) = \frac{f(x+2) - f(x)}{x + 2 - x} = \tfrac{1}{2}[f(x+2) - f(x)].$$

Hence,

$$x^2\big(f(x+2) - f(x)\big) = 2x^2 f'(\tau) = \frac{2x^2}{1 + \tau^2}.$$

Since $0 < x < \tau < x + 2$, it follows that $1 + x^2 < 1 + \tau^2 < 1 + (x+2)^2$. By consequence,

$$\frac{2x^2}{1 + (x+2)^2} < x^2\big(f(x+2) - f(x)\big) < \frac{2x^2}{1 + x^2}.$$

Because the left-hand side and right-hand side of this inequality go to 2 as $x \to \infty$, the limit of the 'sandwiched' expression is also 2.

9.38 Since $f(0) = f'(0) = 0$, the linear approximation ℓ_0 of the function f at 0 is given by

$$\ell_0(x) = f(0) + f'(0)x = 0.$$

So $\ell_0 = 0$. As $f'' = 2$, for $x \in \big(-\frac{1}{2}, \frac{1}{2}\big)$ and $x \neq 0$ the remainder r satisfies

$$r(x) = \frac{f''(\tau)}{2} x^2 = x^2 < \tfrac{1}{4},$$

where τ is between 0 and x.

9.39 (a) As $(x-1)^2 = 0 \iff x = 1$, the domain of th function is $\mathbb{R} \setminus \{1\}$.

(b) For $x \neq 1$,

$$f'(x) = \frac{(x-1)^2 \cdot 1 - x \cdot 2(x-1)}{(x-1)^4} = \frac{x - 1 - 2x}{(x-1)^3} = -\frac{1 + x}{(x-1)^3}$$

and

$$f''(x) = -\frac{(x-1)^3 \cdot 1 - (1+x) \cdot 3(x-1)^2}{(x-1)^6} = -\frac{x - 1 - 3(1+x)}{(x-1)^4} = \frac{2x + 4}{(x-1)^4}.$$

(c) As $f'(x) = 0 \iff x = -1$ and f' switches sign at -1 from negative to positive, the function has a minimum at -1.

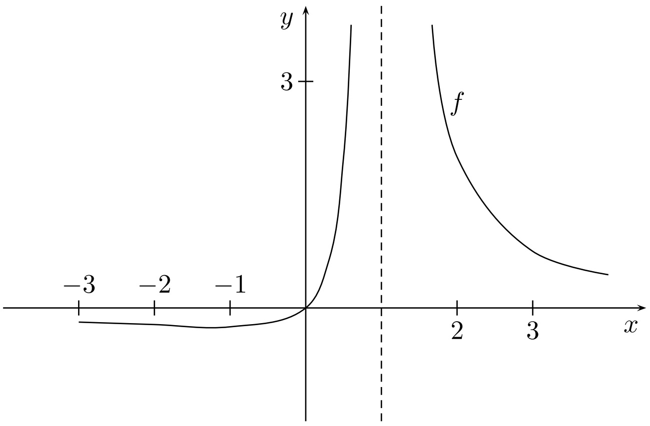
(d) Note that $f''(x) = 0 \iff x = -2$. As $f'' < 0$ on the interval $(-\infty, -2)$, the function is concave on this interval. As $f'' > 0$ on the intervals $(-2, 1)$ and $(1, \infty)$, the function is convex on these intervals.

(e) As, for $x \neq 1$,

$$f(x) = \frac{x}{(x-1)^2} = \frac{x-1+1}{(x-1)^2} = \frac{1}{x-1} + \frac{1}{(x-1)^2},$$

the two limits $\lim_{x\to-\infty} f(x)$ and $\lim_{x\to\infty} f(x)$ are both equal to 0.

(f)



11.16 Note that for $x > 0$

$$\frac{x^3}{x+1} = \frac{x^2(x+1) - x^2}{x+1} = \frac{x^2(x+1) - x(x+1) + x + 1 - 1}{x+1} = x^2 - x + 1 - \frac{1}{x+1}.$$

Hence,

$$\int_0^1 \frac{x^3}{x+1}\,\mathrm{d}x = \int_0^1 \left(x^2 - x + 1 - \frac{1}{x+1}\right)\mathrm{d}x = \left[\tfrac{1}{3}x^3 - \tfrac{1}{2}x^2 + x - \ln(x+1)\right]_0^1$$
$$= \tfrac{1}{3} - \tfrac{1}{2} + 1 - \ln 2 = \tfrac{5}{6} - \ln 2.$$

11.17 In order to evaluate the integral

$$\int_0^1 \frac{x^3}{(1+x^2)^3}\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 \frac{x^2}{(1+x^2)^3} \cdot 2x\,\mathrm{d}x,$$

we use the Method of Substitution with $\varphi(x) = 1 + x^2$. Then

$$\int_0^1 \frac{x^3}{(1+x^2)^3}\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 \frac{x^2}{(1+x^2)^3} \cdot 2x\,\mathrm{d}x = \tfrac{1}{2}\int_0^1 \frac{\varphi(x)-1}{\varphi(x)^3} \cdot \varphi'(x)\,\mathrm{d}x$$
$$= \tfrac{1}{2}\int_0^1 \left(\frac{1}{\varphi(x)^2} - \frac{1}{\varphi(x)^3}\right) \cdot \varphi'(x)\,\mathrm{d}x = \tfrac{1}{2}\left[-\varphi(x)^{-1} + \tfrac{1}{2}\varphi(x)^{-2}\right]_0^1$$
$$= \tfrac{1}{2}\left(-\frac{1}{\varphi(1)} + \frac{1}{2\varphi(1)^2} + \frac{1}{\varphi(0)} - \frac{1}{2\varphi(0)^2}\right) = \tfrac{1}{16}.$$