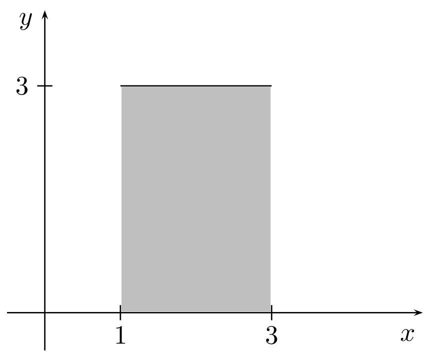
10.1 (a) For $1 < p \le 4$, $A(p)$ denotes the area of the region bounded by the horizontal axis, the vertical lines $x = 1$ and $x = p$ and the graph of the function $g\colon x \to 3$. So the 'area function' is given by

$$A(p) = 3(p-1).$$

(b) For $1 < p \le 5$, $A(p)$ denotes the area of the region bounded by the horizontal axis, the vertical lines $x = 1$ and $x = p$ and the graph of the function $h\colon x \to x$. So the 'area function' is given by

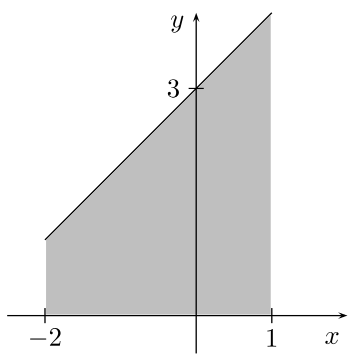
$$A(p) = (p-1) \times 1 + \tfrac{1}{2}(p-1) \times (p-1) = \tfrac{1}{2}p^2 - \tfrac{1}{2}.$$

10.2 (a) The integral $\displaystyle\int_1^3 3\,\mathrm{d}x$ represents the area of the region



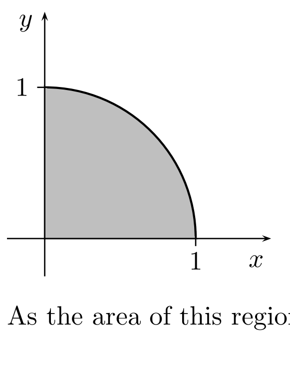
As the area of this region is 6, the given integral is equal to 6.

(b) The integral $\displaystyle\int_{-2}^{1} (t+3)\,\mathrm{d}t$ represents the area of the region



As the area of this region is $7\frac{1}{2}$, the given integral is equal to $7\frac{1}{2}$.

(c) The integral $\displaystyle\int_0^1 \sqrt{1-x^2}\,\mathrm{d}x$ represents the area of the region

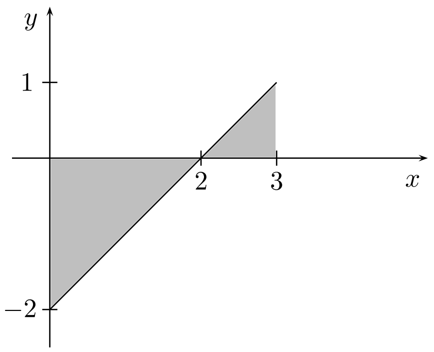


As the area of this region is $\frac{1}{4}\pi$, the given integral is equal to $\frac{1}{4}\pi$.

10.3 (a) The integral

$$\int_0^3 (x-2)\,\mathrm{d}x = \left[\tfrac{1}{2}x^2 - 2x\right]_0^3 = 4\tfrac{1}{2} - 6 = -1\tfrac{1}{2}$$

represents the area $\frac{1}{2}$ of the triangle above the horizontal axis minus the area 2 of the triangle below the horizontal axis.



(b) The integral

$$\int_2^3 (t-2)\,\mathrm{d}t = \left[\tfrac{1}{2}t^2 - 2t\right]_2^3 = 4\tfrac{1}{2} - 6 - 2 + 4 = \tfrac{1}{2}$$

represents the area of the triangle above the horizontal axis (in the figure of part (a)).

10.6 We introduce the continuous function g on the interval $[0,1]$, defined by

$$g(t) = t^2.$$

Then

$$G(x) = \int_0^x f(t^2)\,\mathrm{d}t = \int_0^x f\big(g(t)\big)\,\mathrm{d}t = \int_0^x \big(f \circ g\big)(t)\,\mathrm{d}t$$

and the Fundamental Theorem, part I implies that, for $0 < x < 1$,

$$G'(x) = \big(f \circ g\big)(x) = f\big(g(x)\big) = f(x^2).$$

10.7 We introduce the differentiable functions g and h on the interval $[0,1]$, defined by

$$g(x) = x^2$$

and

$$h(x) = \int_0^x f(t)\,\mathrm{d}t.$$

Then $G(x) = h\big(g(x)\big)$ for all $x \in [0,1]$. In view of the Fundamental Theorem, part I and the Chain Rule, the function G is differentiable and for $0 < x < 1$,

$$G'(x) = h'\big(g(x)\big) \cdot g'(x) = f\big(g(x)\big) \cdot 2x = f(x^2) \cdot 2x.$$

10.10 (a) Let $\varepsilon > 0$. Choose $H = \ln\Big(\dfrac{2}{\varepsilon}\Big)^2$. Then

$$t > H \Longrightarrow \mathrm{e}^t > \mathrm{e}^H = \Big(\frac{2}{\varepsilon}\Big)^2 \Longrightarrow \sqrt{\mathrm{e}^t} > \frac{2}{\varepsilon} \Longrightarrow \frac{1}{\sqrt{\mathrm{e}^t}} < \frac{\varepsilon}{2} \Longrightarrow \frac{2}{\sqrt{\mathrm{e}^t}} < \varepsilon \Longrightarrow \Big|\frac{2}{\sqrt{\mathrm{e}^t}} - 0\Big| < \varepsilon.$$

This proves that $\displaystyle\lim_{t\to\infty} \frac{2}{\sqrt{\mathrm{e}^t}} = 0$.

(b) Let $\varepsilon > 0$. Note that for all $x \geq 0$,

$$0 < \frac{x}{\mathrm{e}^x} < \frac{2}{\sqrt{\mathrm{e}^x}}.$$

So for $x > H$, $0 < \dfrac{x}{\mathrm{e}^x} < \dfrac{2}{\sqrt{\mathrm{e}^x}} < \varepsilon$, which implies that $\left|\dfrac{x}{\mathrm{e}^x} - 0\right| < \varepsilon$.

This proves that $\lim\limits_{x\to\infty} \dfrac{x}{\mathrm{e}^x} = 0$.

10.12 (a) According to the Arithmetic Rules for limits

$$\lim_{x\to\infty} \frac{2+\mathrm{e}^x}{1+3\,\mathrm{e}^x} = \lim_{x\to\infty} \frac{1+2\,\mathrm{e}^{-x}}{3+\mathrm{e}^{-x}} = \tfrac{1}{3}.$$

Note that $\lim\limits_{x\to\infty} [1+2\,\mathrm{e}^{-x}] = 1$ and $\lim\limits_{x\to\infty} [3+\mathrm{e}^{-x}] = 3$.

(b) If $f(x) = \mathrm{e}^{2x} - \mathrm{e}^x$ and $g(x) = x$, then the functions f and g are differentiable, $g'(x) = 1 \neq 0$ for all x and $f(0) = g(0) = 0$. Furthermore, $f'(x) = 2\,\mathrm{e}^{2x} - \mathrm{e}^x$. So, in view of de l'Hôpital's Rule (weak form),

$$\lim_{x\to 0} \frac{\mathrm{e}^{2x} - \mathrm{e}^x}{x} = \lim_{x\to 0} \frac{f(x)}{g(x)} = \frac{f'(0)}{g'(0)} = 1.$$

11.8 (a) For $b > 1$

$$\int_1^b x\,\mathrm{e}^{-x}\,\mathrm{d}x = \left[-x\,\mathrm{e}^{-x}\right]_1^b - \int_1^b -\mathrm{e}^{-x}\,\mathrm{d}x = -b\,\mathrm{e}^{-b} + \mathrm{e}^{-1} - \left[\mathrm{e}^{-x}\right]_1^b$$

$$= -b\,\mathrm{e}^{-b} + \mathrm{e}^{-1} - \mathrm{e}^{-b} + \mathrm{e}^{-1} = \frac{2}{\mathrm{e}} - \mathrm{e}^{-b} - b\,\mathrm{e}^{-b}.$$

Because $\lim\limits_{b\to\infty} \left(\dfrac{2}{\mathrm{e}} - \mathrm{e}^{-b} - b\,\mathrm{e}^{-b}\right) = \dfrac{2}{\mathrm{e}}$,

$$\int_1^\infty x\,\mathrm{e}^{-x}\,\mathrm{d}x = \frac{2}{\mathrm{e}}.$$

(b) For $0 < b < 1$ and with $\varphi(x) = 1 - x$

$$\int_0^b \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = \int_0^b -\frac{1}{\sqrt{\varphi(x)}} \cdot \varphi'(x)\,\mathrm{d}x = \left[-2\sqrt{\varphi(x)}\right]_0^b = \left[-2\sqrt{1-x}\right]_0^b = -2\sqrt{1-b} + 2.$$

Because $\lim\limits_{b\to 1} \left(-2\sqrt{1-b} + 2\right) = 2$,

$$\int_0^1 \frac{1}{\sqrt{1-x}}\,\mathrm{d}x = 2.$$

11.10 As, for $b > 0$,

$$\int_0^b \frac{1}{1+x^2}\,\mathrm{d}x = [\arctan x]_0^b = \arctan b$$

and $\lim\limits_{b\to\infty} \arctan b = \tfrac{1}{2}\pi$,

$$\int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\pi.$$

Similarly,

$$\int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{a\to-\infty} \int_a^0 \frac{1}{1+x^2}\,\mathrm{d}x = \lim_{a\to-\infty} [\arctan x]_a^0 = -\lim_{a\to-\infty} \arctan a = \tfrac{1}{2}\pi.$$

Hence,

$$\int_{-\infty}^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \int_{-\infty}^0 \frac{1}{1+x^2}\,\mathrm{d}x + \int_0^\infty \frac{1}{1+x^2}\,\mathrm{d}x = \tfrac{1}{2}\pi + \tfrac{1}{2}\pi = \pi.$$

11.18 Note that

$$\int_0^3 x\sqrt{1+x}\,\mathrm{d}x = \int_0^3 \left((1+x)\sqrt{1+x} - \sqrt{1+x}\right)\,\mathrm{d}x = \int_0^3 \left((1+x)^{1\frac{1}{2}} - (1+x)^{\frac{1}{2}}\right)\,\mathrm{d}x$$

$$= \left[\tfrac{2}{5}(1+x)^{2\frac{1}{2}} - \tfrac{2}{3}(1+x)^{1\frac{1}{2}}\right]_0^3 = \tfrac{64}{5} - \tfrac{16}{3} - \tfrac{2}{5} + \tfrac{2}{3} = \tfrac{116}{15} = 7\tfrac{11}{15}.$$