10.4 We assume that $u > 0$ and $\ell < 0$.

The integral $\int_a^b f(x)\,\mathrm{d}x$ is the area of the region above the horizontal axis minus the area of the region below the horizontal axis.

The area above the horizontal axis is contained in the rectangle 'above' the interval $[a, b]$ and below the line at height u. The area of this rectangle is $u(b-a)$. So the area of the region above the horizontal axis is at most equal to $u(b-a)$. As a consequence, the integral is at most equal to $u(b-a)$.

Similarly, the area of the region below the horizontal axis is at most $-\ell(b-a)$. So the integral is at least equal to $-[-\ell(b-a)] = \ell(b-a)$.

10.8 The function G on $[a, b]$ defined by

$$G(x) = \int_a^x f(t)\,\mathrm{d}t$$

is continuous on $[a, b]$ and differentiable on (a, b). According to the Mean Value Theorem there exists a $\tau \in (a, b)$ such that

$$\frac{G(b) - G(a)}{b-a} = G'(\tau) = f(\tau) \Longrightarrow \frac{1}{b-a} G(b) = f(\tau) \Longrightarrow \frac{1}{b-a} \int_a^b f(x)\,\mathrm{d}x = f(\tau).$$

10.9 (a) Let h be the function on $[a, b]$ defined by $h = g - f$. Then h is a non-negative function. So according to Theorem 3,

$$0 \leq \int_a^b h(x)\,\mathrm{d}x = \int_a^b [g(x) - f(x)]\,\mathrm{d}x = \int_a^b g(x)\,\mathrm{d}x - \int_a^b f(x)\,\mathrm{d}x.$$

Hence, $\int_a^b f(x)\,\mathrm{d}x \leq \int_a^b g(x)\,\mathrm{d}x$.

(b) If f is a continuous function on an interval $[a, b]$, then

$$-|f(x)| \leq f(x) \leq |f(x)|,$$

for all $x \in [a, b]$. Since $|f|$ is a continuous function on the interval $[a, b]$, part (a) implies that

$$\int_a^b -|f(x)|\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x \Longleftrightarrow -\int_a^b |f(x)|\,\mathrm{d}x \leq \int_a^b f(x)\,\mathrm{d}x \leq \int_a^b |f(x)|\,\mathrm{d}x$$

$$\Longleftrightarrow \left| \int_a^b f(x)\,\mathrm{d}x \right| \leq \int_a^b |f(x)|\,\mathrm{d}x.$$

10.11 (a) As the functions $x \mapsto |x|$ and $x \mapsto \mathrm{e}^x$ are continuous, their composition is continuous at 0.

(b) Let $g(x) = \mathrm{e}^x$ and $h(x) = \mathrm{e}^{-x}$. Then

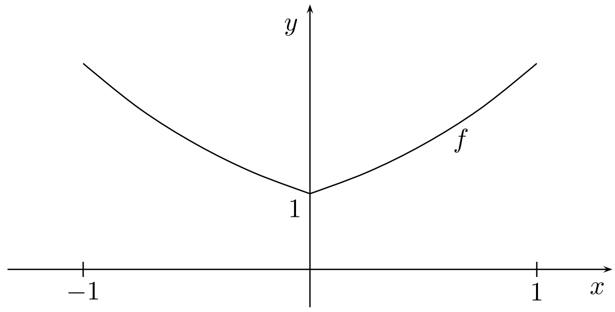
$$\lim_{x \downarrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \downarrow 0} \frac{\mathrm{e}^x - 1}{x} = g'(0) = 1$$

and

$$\lim_{x \uparrow 0} \frac{f(x) - f(0)}{x} = \lim_{x \uparrow 0} \frac{\mathrm{e}^{-x} - 1}{x} = h'(0) = -1.$$

As the right-hand derivative of f at 0 is different from the left-hand derivative of f at 0, the function f is not differentiable at 0.

(c) The graph of f is presented in the following figure.



11.8 (c) For $b > 4$

$$\int_4^b \frac{\ln x}{x^2}\,\mathrm{d}x = \left[-\frac{\ln x}{x}\right]_4^b + \int_4^b \frac{1}{x^2}\,\mathrm{d}x = -\frac{\ln b}{b} + \frac{\ln 4}{4} - \left[\frac{1}{x}\right]_4^b = -\frac{\ln b}{b} + \frac{\ln 4}{4} - \frac{1}{b} + \frac{1}{4}.$$

Because $\lim\limits_{b\to\infty} \left(-\frac{\ln b}{b} + \frac{\ln 4}{4} - \frac{1}{b} + \frac{1}{4}\right) = \frac{1}{4}\left(1 + \ln 4\right)$,

$$\int_4^\infty \frac{\ln x}{x^2}\,\mathrm{d}x = \tfrac{1}{4}\left(1 + \ln 4\right).$$

(d) For $b > 3$ and with $\varphi(x) = \ln x$,

$$\int_3^b \frac{1}{x\,\ln x}\,\mathrm{d}x = \int_3^b \frac{1}{\ln x}\cdot\frac{1}{x}\,\mathrm{d}x = \int_3^b \frac{1}{\varphi(x)}\cdot\varphi'(x)\,\mathrm{d}x$$
$$= [\ln\varphi(x)]_3^b = [\ln(\ln x)]_3^b = \ln(\ln b) - \ln(\ln 3).$$

Because $\ln(\ln b) \to \infty$ as $b \to \infty$, the integral is a divergent one.

11.11 If the integral $\int_{-\infty}^{\infty} f(x)\,\mathrm{d}x$ exists, the limits $\lim\limits_{b\to-\infty}\int_b^0 f(x)\,\mathrm{d}x$ and $\lim\limits_{b\to\infty}\int_0^b f(x)\,\mathrm{d}x$ exist. Since

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = \int_{-a}^{0} f(x)\,\mathrm{d}x + \int_0^a f(x)\,\mathrm{d}x,$$

and the limits (as $a \to \infty$) of the two expressions at the right-hand side of the equality sign exist, the Arithmetic Rules for limits of functions imply that the limit (as $a \to \infty$) of the expression at the left-hand side of the equality sign exists.

11.19 We consider for $b > 0$ the integral

$$\int_0^b \frac{4x}{1+x^4}\,\mathrm{d}x = 2\int_0^b \frac{1}{1+\left(x^2\right)^2}\cdot 2x\,\mathrm{d}x.$$

In order to evaluate this integral we use the Method of Substitution with $\varphi(x) = x^2$. Then

$$\int_0^b \frac{4x}{1+x^4}\,\mathrm{d}x = 2\int_0^b \frac{1}{1+\left(x^2\right)^2}\cdot 2x\,\mathrm{d}x = 2\int_0^b \frac{1}{1+\varphi(x)^2}\cdot\varphi'(x)\,\mathrm{d}x$$
$$= 2\left[\arctan\left(\varphi(x)\right)\right]_0^b = 2\arctan\left(\varphi(b)\right) - 2\arctan\left(\varphi(0)\right)$$
$$= 2\arctan\left(b^2\right) - 2\arctan(0) = 2\arctan\left(b^2\right).$$

As $\lim\limits_{b\to\infty}\arctan\left(b^2\right) = \frac{1}{2}\pi$, the given integral is equal to π.