1.3 We have to show that (a) $\big(A \setminus B\big) \cup \big(A \cap B\big) \subset A$ and that (b) $A \subset \big(A \setminus B\big) \cup \big(A \cap B\big)$.

(a) Let $x \in \big(A \setminus B\big) \cup \big(A \cap B\big)$. Then $x \in A \setminus B$ or $x \in A \cap B$. If $x \in A \setminus B$, then $x \in A$ and if $x \in A \cap B$, then $x \in A$ too. So $x \in A$.

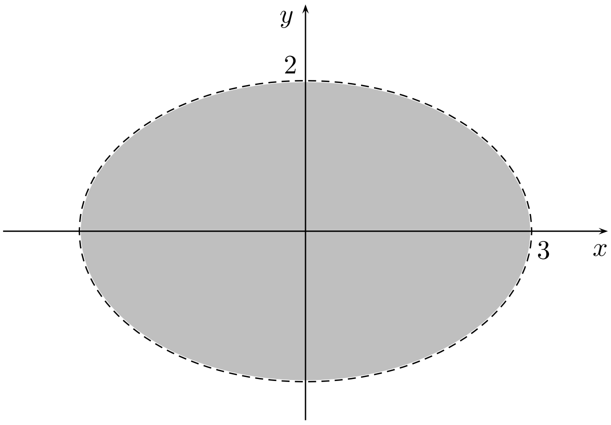
(b) Let $x \in A$. We distinguish two cases: $x \in B$ and $x \notin B$.

If $x \in B$, then $x \in A \cap B$. If $x \notin B$, then $x \in A \setminus B$. Hence, x is contained in $A \cap B$ or x is contained in $A \setminus B$, i.e. $x \in \big(A \setminus B\big) \cup \big(A \cap B\big)$.

1.8 (g) Note that

$$4x^2 + 9y^2 = 36 \Longleftrightarrow \frac{x^2}{9} + \frac{y^2}{4} = 1$$

represents an *ellipse* which passes through the four points $(3, 0)$, $(-3, 0)$, $(0, 2)$ and $(0, -2)$. This ellipse is a circle that has been squashed by scaling it by different amounts in the two coordinate directions. The set $4x^2 + 9y^2 < 36$ represents the area enclosed by this ellipse:



(h) Note that

$$x^2 - y^2 = 4 \Longleftrightarrow y^2 = x^2 - 4.$$

Hence, $x^2 - 4 \geq 0 \Longrightarrow x \leq -2$ or $x \geq 2$. For these values of x,

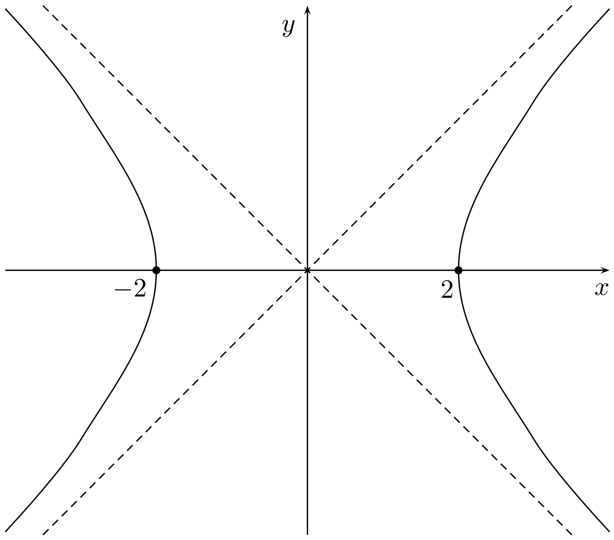
$$y^2 = x^2 - 4 \Longleftrightarrow y = \pm\sqrt{x^2 - 4}.$$

So the curve is in two parts called *branches* (one part for $x \geq 2$ and one part for $x \leq -2$). As for large values of x,

$$\pm\sqrt{x^2 - 4} = \pm x\sqrt{1 - \frac{4}{x^2}} \approx \pm x,$$

the branches are (for large values of x) close to the straight lines $y = x$ and $y = -x$. These lines, which intersect at right angles, are called *asymptotes*. The same holds if $-x$ is large.

The curve is called a (*rectangular*) *hyperbola* that has center at the origin and that passes through the points $(2, 0)$ and $(-2, 0)$. The adjective rectangular refers to the fact that the two asymptotes intersect at right angles.



1.26 (a) For $x \neq 1$ and $g(x) \neq 0$,

$$(f \circ g)(x) = f\big(g(x)\big) = f\Big(\frac{x}{1-x}\Big) = \frac{2}{\dfrac{x}{1-x}} = \frac{2-2x}{x} = \frac{2}{x} - 2.$$

As $g(x) = 0 \iff x = 0$, the domain of $f \circ g$ is $\mathbb{R} \setminus \{0, 1\}$.

For $x \neq 0$ and $f(x) \neq 1$,

$$(g \circ f)(x) = g\big(f(x)\big) = g\Big(\frac{2}{x}\Big) = \frac{\dfrac{2}{x}}{1 - \dfrac{2}{x}} = \frac{2}{x-2}.$$

As $f(x) = 1 \iff x = 2$, the domain of $g \circ f$ is $\mathbb{R} \setminus \{0, 2\}$.

For $x \neq 0$ and $f(x) \neq 0$,

$$(f \circ f)(x) = f\big(f(x)\big) = f\Big(\frac{2}{x}\Big) = \frac{2}{\dfrac{2}{x}} = x.$$

As $f(x) \neq 0$ for all $x \neq 0$, the domain of $f \circ f$ is $\mathbb{R} \setminus \{0\}$.

For $x \neq 1$ and $g(x) \neq 1$,

$$(g \circ g)(x) = g\big(g(x)\big) = g\Big(\frac{x}{1-x}\Big) = \frac{\dfrac{x}{1-x}}{1 - \dfrac{x}{1-x}} = \frac{x}{1-2x}.$$

As $g(x) = 1 \iff x = \frac{1}{2}$, the domain of $g \circ g$ is $\mathbb{R} \setminus \{1, \frac{1}{2}\}$.

1.27 (a) $(f \circ g)(x) = f\big(g(x)\big) = g(x)^2 = (x+1)^2$.

(b) As $x = (f \circ g)(x) = f(x+4)$, it follows that $f(x) = x - 4$.

(c) As $|x| = (f \circ g)(x) = f\big(g(x)\big) = \sqrt{g(x)}$, it follows that $g(x) = x^2$.

(d) As $2x + 3 = (f \circ g)(x) = f\big(g(x)\big) = f\big(\sqrt[3]{x}\big)$, it follows that $f(x) = 2x^3 + 3$.

(e) As

$$x = (f \circ g)(x) = f\big(g(x)\big) = \frac{g(x)+1}{g(x)},$$

provided that $g(x) \neq 0$, it follows that

$$g(x)x = g(x) + 1 \Longrightarrow g(x)(x-1) = 1 \Longrightarrow g(x) = \frac{1}{x-1},$$

provided that $x \neq 1$.

(f) As $\dfrac{1}{x^2} = (f \circ g)(x) = f\big(g(x)\big) = f(x-1)$, it follows that

$$f(x) = f\big((x+1) - 1\big) = \frac{1}{(x+1)^2},$$

provided that $x \neq -1$.

1.49 Let h and g be the functions defined by $h(x) = x^2$ and $g(x) = \dfrac{e^x + e^{-x}}{x+1}$ $(x \neq -1)$, respectively. Then for all x,

$$(g \circ h)(x) = g\big(h(x)\big) = \frac{e^{h(x)} + e^{-h(x)}}{h(x)+1} = \frac{e^{x^2} + e^{-x^2}}{x^2+1} = f(x).$$

So $f = g \circ h$.

1.51 (c) $x^3y - 4x^2y^2 + 4xy^3 = xy(x^2 - 4xy + 4y^2) = xy(x-2y)^2$.

(d) $x^2 + 2xy - 3y^2 = (x+3y)(x-y)$.

(e) For this exercise some inventiveness is needed: first rearrange the terms of the sum.

$$x^3 + y^3 + xy^2 + x^2y = x^3 + x^2y + y^3 + xy^2 = x^2(x+y) + y^2(y+x) = (x+y)(x^2+y^2).$$

1.52 (d) Using the fact that all the terms in the fraction have the factor $\sqrt{x}\sqrt{y}$ in common, we obtain

$$\frac{x\sqrt{y} - y\sqrt{x}}{x\sqrt{y} + y\sqrt{x}} = \frac{\sqrt{x}\sqrt{y}\big[\sqrt{x} - \sqrt{y}\big]}{\sqrt{x}\sqrt{y}\big[\sqrt{x} + \sqrt{y}\big]} = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}.$$

By multiplying the numerator and denominator by $x\sqrt{y} - y\sqrt{x}$ we obtain

$$\frac{x\sqrt{y} - y\sqrt{x}}{x\sqrt{y} + y\sqrt{x}} = \frac{\big[x\sqrt{y} - y\sqrt{x}\big]^2}{\big[x\sqrt{y} + y\sqrt{x}\big]\big[x\sqrt{y} - y\sqrt{x}\big]} = \frac{x^2y - 2xy\sqrt{xy} + y^2x}{\big[x\sqrt{y}\big]^2 - \big[y\sqrt{x}\big]^2} = \frac{x^2y - 2xy\sqrt{xy} + y^2x}{x^2y - y^2x}$$

$$= \frac{xy\big[x - 2\sqrt{xy} + y\big]}{xy\big[x-y\big]} = \frac{\big[\sqrt{x} - \sqrt{y}\big]^2}{x-y} = \frac{\big[\sqrt{x} - \sqrt{y}\big]^2}{\big[\sqrt{x} - \sqrt{y}\big]\big[\sqrt{x} + \sqrt{y}\big]} = \frac{\sqrt{x} - \sqrt{y}}{\sqrt{x} + \sqrt{y}}.$$

1.53 (a) First we solve the corresponding equality.

$$x^4 - 5x^2 + 5 = 1 \Longleftrightarrow x^4 - 5x^2 + 4 = 0 \Longleftrightarrow (x^2-4)(x^2-1) = 0$$

$$\Longleftrightarrow (x-2)(x+2)(x-1)(x+1) = 0 \Longleftrightarrow x = \pm 1 \text{ or } x = \pm 2.$$

Next we construct a *sign diagram* for the expression $x^4 - 5x^2 + 4$:

+	0	−	0	+	0	−	0	+
	−2		−1		1		2	

So the inequality is satisfied for $x \leq -2$ or $-1 \leq x \leq 1$ or $x \geq 2$.