1.10 (a) We distinguish three cases: $x < 0$, $0 \leq x \leq 3$ and $x > 3$.

If $x < 0$,

$$|x-3| = 2|x| \Longleftrightarrow 3 - x = -2x \Longleftrightarrow x = -3.$$

If $0 \leq x \leq 3$,

$$|x-3| = 2|x| \Longleftrightarrow 3 - x = 2x \Longleftrightarrow x = 1.$$

If $x > 3$,

$$|x-3| = 2|x| \Longleftrightarrow x - 3 = 2x \Longleftrightarrow x = -3.$$

This doesn't generate a solution, because -3 is not larger than 3.

So the solutions are $x = 1$ and $x = -3$.

(b) Since $x^2 - 1 \geq 0 \Longleftrightarrow x \leq -1$ or $x \geq 1$, we distinguish the cases $x \in (-1, 1)$ and $x \notin (-1, 1)$.

If $x \in (-1, 1)$, $|x^2 - 1| = 1 - x^2$, so

$$|x^2 - 1| = 2x \Longleftrightarrow 1 - x^2 = 2x \Longleftrightarrow x^2 + 2x - 1 = 0 \Longleftrightarrow x = \frac{-2 \pm \sqrt{4+4}}{2} = -1 \pm \sqrt{2}.$$

Since $-1 - \sqrt{2} \notin (-1, 1)$, we find the solution $x = -1 + \sqrt{2}$.

If $x \notin (-1, 1)$, $|x^2 - 1| = x^2 - 1$, so

$$|x^2 - 1| = 2x \Longleftrightarrow x^2 - 1 = 2x \Longleftrightarrow x^2 - 2x - 1 = 0 \Longleftrightarrow x = \frac{2 \pm \sqrt{4+4}}{2} = 1 \pm \sqrt{2}.$$

Since $1 - \sqrt{2} \in (-1, 1)$, we find the solution $x = 1 + \sqrt{2}$.

1.11 We distinguish the cases $x \geq 0$ and $x < 0$.

If $x \geq 0$, then

$$|x| = x \geq 0 \geq -|x|.$$

If $x < 0$, then

$$-|x| = -(-x) = x < 0 \leq |x|.$$

1.12 Let x and y be two real numbers. We distinguish two cases: $x < y$ and $x \geq y$.

If $x < y$, then $\max\{x, y\} = y$ and

$$\frac{x + y + |y - x|}{2} = \frac{x + y + (y - x)}{2} = y.$$

If $x \geq y$, then $\max\{x, y\} = x$ and

$$\frac{x + y + |y - x|}{2} = \frac{x + y - (y - x)}{2} = x.$$

So in both cases $\max\{x, y\} = \dfrac{x + y + |y - x|}{2}$.

1.14 (a) If $x \geq 0$, then $|x| = x$ and $-x \leq 0$. So $|-x| = -(-x) = x$.

If $x < 0$, then $|x| = -x$ and $-x > 0$. So $|-x| = -x$.

(b) If $x \geq 0$ and $y \geq 0$, then $xy \geq 0$. So $|xy| = xy = x \cdot y = |x|\,|y|$.

If $x \geq 0$ and $y < 0$, then $xy \leq 0$. So $|xy| = -xy = x \cdot (-y) = |x|\,|y|$.

If $x < 0$ and $y \geq 0$, then $xy \leq 0$. So $|xy| = -xy = (-x) \cdot y = |x|\,|y|$.

If $x < 0$ and $y < 0$, then $xy > 0$. So $|xy| = xy = (-x) \cdot (-y) = |x|\,|y|$.

1.16 Let a and b be real numbers. Then

$$a^2 + b^2 - 2|a||b| = |a|^2 + |b|^2 - 2|a||b| = \big(|a| - |b|\big)^2 \geq 0.$$

1.19 (a) Let $x, y \in \mathbb{R}$. Then, according to the Triangle Inequality,

$$|x - y| = |x + (-y)| \leq |x| + |-y| \underset{\text{Theorem 2(a)}}{=} |x| + |y|.$$

(b) Let $x, y, z \in \mathbb{R}$. Then, according to the Triangle Inequality,

$$|x + y + z| = |x + (y + z)| \leq |x| + |y + z| \leq |x| + |y| + |z|.$$

1.20 According to the Triangle Inequality, for real numbers a, b and c,

$$|a - b| = |a - c + c - b| = |(a - c) + (c - b)| \leq |a - c| + |c - b|.$$

1.21 According to the Triangle Inequality (or more precisely Exercise 19 (b)),

$$|4x^2 - 2x + 2| \leq |4x^2| + |-2x| + |2| = 4x^2 + 2|x| + 2.$$

Since for $-1 \leq x < 2$, $x^2 < 4$ and $|x| < 2$,

$$|4x^2 - 2x + 2| \leq 4x^2 + 2|x| + 2 < 16 + 4 + 2 = 22.$$

So we may choose $b = 22$.

Since $|4x^2 - 2x + 2| < 22 \iff -22 < 4x^2 - 2x + 2 < 22$, the result can be formulated as follows. The curve $y = 4x^2 - 2x + 2$ lies between the horizontal lines $y = -22$ and $y = 22$.

1.23 Note that for $2 < x < 4$,

$$\left|\frac{1}{4x^2 - 2x - 2}\right| = \frac{1}{|4x^2 - 2x - 2|}.$$

We will determine a number $\ell > 0$ such that $|4x^2 - 2x - 2| \geq \ell$. For obvious reasons, this number ℓ is called a *lower bound* for the expression $|4x^2 - 2x - 2|$.

According to the Reverse Triangle Inequality,

$$|4x^2 - 2x - 2| = |4x^2 - 2(x + 1)| \geq \big||4x^2| - |2(x + 1)|\big| = \big|4x^2 - 2|x + 1|\big|.$$

Now for $2 < x < 4$, $4x^2 \geq 16$ and $3 < x+1 < 5$, which implies that $|x+1| \leq 5$. Hence, $4x^2 - 2|x+1| > 0$. Therefore

$$|4x^2 - 2x - 2| = \left|4x^2 - 2|x+1|\right| = 4x^2 - 2|x+1| \geq 16 - 10 = 6.$$

So

$$\left|\frac{1}{4x^2 - 2x - 2}\right| \leq \tfrac{1}{6}.$$

1.24 (a) According to Theorem 2 (c),

$$\tfrac{1}{2}|5x-3| < 4 \Longleftrightarrow |5x-3| < 8 \Longleftrightarrow -8 < 5x-3 < 8 \Longleftrightarrow -5 < 5x < 11 \Longleftrightarrow -1 < x < \tfrac{11}{5}.$$

(b) Since $x| \geq 0$ and $|x-1| \geq 0$, $|x| + |x-1| \geq 0$. So

$$|x| + |x-1| \leq 0 \Longrightarrow |x| + |x-1| = 0 \Longrightarrow |x| = 0 \text{ and } |x-1| = 0 \Longrightarrow x = 0 \text{ and } x - 1.$$

Hence, the solution set of this inequality is empty.

(c) According to the observation preceding this exercise,

$$|x-3| > 2|x| \Longleftrightarrow (x-3)^2 > 4x^2 \Longleftrightarrow x^2 - 6x + 9 > 4x^2 \Longleftrightarrow 3x^2 + 6x - 9 < 0$$

$$\Longleftrightarrow 3(x+3)(x-1) < 0 \Longleftrightarrow x \in (-3, 1).$$

(d) We distinguish three cases: $x < 1$, $1 \leq x < 2$ and $x \geq 2$.

If $x < 1$,

$$|x-1| + |x-2| > 1 \Longleftrightarrow 1 - x + 2 - x > 1 \Longleftrightarrow 2x < 2 \Longleftrightarrow x < 1.$$

So the inequality holds for all $x < 1$.

If $1 \leq x < 2$,

$$|x-1| + |x-2| > 1 \Longleftrightarrow x - 1 + 2 - x > 1 \Longleftrightarrow 1 > 1.$$

So the inequality doesn't hold for all $1 \leq x < 2$.

If $x \geq 2$,

$$|x-1| + |x-2| > 1 \Longleftrightarrow x - 1 + x - 2 > 1 \Longleftrightarrow 2x > 4 \Longleftrightarrow x > 2.$$

So the inequality holds for all $x > 2$.

Hence, the solution set for the original inequality is $(-\infty, 1) \cup (2, \infty)$.

1.28 For $x_1 \neq x_2$,

$$\frac{\ell(x_2) - \ell(x_1)}{x_2 - x_1} = \frac{[mx_2 + b] - [mx_1 + b]}{x_2 - x_1} = \frac{m(x_2 - x_1)}{x_2 - x_1} = m.$$