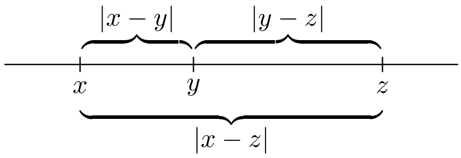
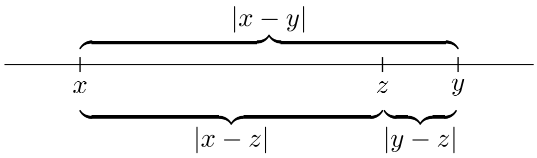
1.13 (a) The result can be formulated as follows. If $y$ is in the interval $(x, z)$, then the distance between $x$ and $z$ is equal to the distance between $x$ and $y$ plus the distance between $y$ and $z$.




If $x < y < z$, then

$$|x - y| + |y - z| = y - x + z - y = z - x = |x - z|.$$

(b) The result can be formulated as follows. If $y$ is outside the interval $(x, z)$, then the distance between $x$ and $z$ is smaller than the distance between $x$ and $y$ plus the distance between $y$ and $z$.




Observe that $x < z < y$ implies that $x < y$. Hence,

$$|x - y| + |y - z| = y - x + y - z = 2y - x - z$$
$$|x - z| = z - x.$$

After these preparations we give a proof by contradiction. If $|x - y| + |y - z| = |x - z|$, then

$$2y - x - z = z - x \Longrightarrow y = z$$

This however contradicts the fact that $y > z$.

**Alternative:**

As $z$ is between $x$ and $y$, part (a) implies that

$$|x - z| + |z - y| = |x - y|.$$

So

$$|x - y| + |y - z| = |x - z| + 2|z - y| > |x - z|.$$

Note that $|z - y| > 0$, because $z \neq y$.

1.15 (a) Let $a \geq 0$ and $b \geq 0$.

We distinguish two cases: $a = b = 0$ and $a \neq 0$ or $b \neq 0$.

In the first case the equivalence is obviously true.

In the second case $a + b > 0$ so that

$$a = b \Longleftrightarrow a - b = 0 \Longleftrightarrow (a - b)\underbrace{(a + b)}_{>0} = 0 \Longleftrightarrow a^2 - b^2 = 0 \Longleftrightarrow a^2 = b^2.$$

(b) Using part (a) and the property $|x|^2 = x^2$, we obtain

$$|x - 3| = 2|x| \Longleftrightarrow (x - 3)^2 = 4x^2 \Longleftrightarrow x^2 - 6x + 9 = 4x^2 \Longleftrightarrow 3x^2 + 6x - 9 = 0$$

$$\Longleftrightarrow x^2 + 2x - 3 = 0 \Longleftrightarrow (x - 1)(x + 3) = 0.$$

So the solutions are $x = -3$ and $x = 1$.

(c) Using part (a) leads to

$$|x+y| = |x|+|y| \iff (x+y)^2 = (|x|+|y|)^2 \iff x^2+y^2+2xy = x^2+y^2+2|x||y| \iff xy = |xy|$$

$$\iff xy \geq 0.$$

1.17 Let $x$ be a real number between $-1$ and 1. Since the inequalities hold for $x = 0$, we may assume that $x \neq 0$. Hence $|x| > 0$.

(a) According to Theorem 2 (c), $|x| < 1$. Hence, Theorem 2 (b) and the fact that $|x| > 0$ imply

$$x^2 = |x^2| = |x| \cdot |x| < |x|.$$

Here the last inequality is obtained by multiplying both sides of the inequality $|x| < 1$ by $|x| > 0$.

(b) Theorem 2 (b) and the fact that $x^2 < 1$ imply

$$\left|x^3\right| = \left|x^2 \cdot x\right| = |x^2| \cdot |x| = x^2|x| < |x|.$$

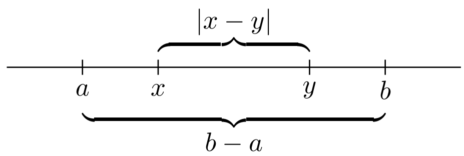
Here the last inequality is obtained by multiplying both sides of the inequality $x^2 < 1$ by $|x| > 0$.

**Alternative**

By using part (a) twice, one obtains

$$\left|x^3\right| = \left|x^2 \cdot x\right| = |x^2| \cdot |x| = x^2|x| < |x||x| = x^2 < |x|.$$

1.18 The result can be formulated as follows. The distance between two points in the interval $(a, b)$ is smaller than the length of this interval:




Note that, according to Theorem 2,

$$|x-y| < b-a \iff a-b < x-y < b-a.$$

The inequality $a - b < x - y$ can be proved as follows:

$$y < b \Longrightarrow \quad \begin{array}{rcl} x & > & a \\ -y & > & -b \\ \hline x-y & > & a-b \end{array} \; +$$

The inequality $x - y < b - a$ can be proved as follows:

$$y > a \Longrightarrow \quad \begin{array}{rcl} x & < & b \\ -y & < & -a \\ \hline x-y & < & b-a \end{array} \; +$$

1.22 Let $-1 < x < 1$. According to the Triangle Inequality,

$$|x^2(1+2x)| = |x^2+2x^3| \leq |x^2| + |2x^3| = x^2 + 2|x^3|.$$

In view of Exercise 17, $x^2 \leq |x|$ and $|x^3| \leq |x|$.

As a consequence $|x^2(1+2x)| \leq x^2 + 2|x^3| \leq 3|x|$. So we may choose $b = 3$.

1.24 (e) We distinguish two cases: $x \in (-1,1)$ and $x \notin (-1,1)$.

If $x \in (-1,1)$,

$$|x^2-1| \leq 2x-2 \Longleftrightarrow 1-x^2 \leq 2x-2 \Longleftrightarrow x^2+2x-3 \geq 0$$

$$\Longleftrightarrow (x+3)(x-1) \geq 0 \Longleftrightarrow x < -3 \text{ or } x > 1.$$

So the inequality doesn't hold for any $x \in (-1,1)$.

If $x \notin (-1,1)$,

$$|x^2-1| \leq 2x-2 \Longleftrightarrow x^2-1 \leq 2x-2 \Longleftrightarrow x^2-2x+1 \leq 0 \Longleftrightarrow (x-1)^2 \leq 0 \Longleftrightarrow x = 1.$$

So the inequality holds for $x = 1$.

**Alternative**

As $2x-2 \geq |x^2-1| \geq 0$, it follows that $x \geq 1$. In that case, $x^2-1 \geq 0$, so that $|x^2-1| = x^2-1$.

As a consequence,

$$|x^2-1| \leq 2x-2 \Longleftrightarrow x^2-1 \leq 2x-2 \Longleftrightarrow x^2-2x+1 \leq 0 \Longleftrightarrow (x-1)^2 \leq 0 \Longleftrightarrow x = 1.$$

(f) Observe that

$$|x-1| \cdot |x+2| \geq 4 \Longleftrightarrow |(x-1)(x+2)| \geq 4.$$

We distinguish two cases: $x \in (-2,1)$ and $x \notin (-2,1)$.

If $x \in (-2,1)$,

$$|(x-1)(x+2)| \geq 4 \Longleftrightarrow -(x-1)(x+2) \geq 4 \Longleftrightarrow x^2+x+2 \leq 0.$$

So the inequality doesn't hold for any $x \in (-2,1)$.

If $x \notin (-2,1)$,

$$|(x-1)(x+2)| \geq 4 \Longleftrightarrow (x-1)(x+2) \geq 4 \Longleftrightarrow x^2+x-6 \geq 0 \Longleftrightarrow (x+3)(x-2) \geq 0.$$

So the inequality holds for all $x \notin (-3,2)$.

Hence, the solution set for the original inequality is $(-\infty,-3] \cup [2,\infty)$.

1.29 (a) Such an equation is

$$y-1 = 2(x-3) \Longleftrightarrow y = 2x-5.$$

(b) An equation of a (non-vertical) line through the point $(1,1)$ is

$$y-1 = m(x-1),$$

where $m$ is some real number. This line contains the point $(2,3)$ if and only if

$$3-1 = m(2-1) \Longleftrightarrow m = 2.$$

So we obtain the equation $y = 2(x-1)+1$ or $y = 2x-1$.

(c) If $x_1 = x_2$, we obtain the equation $x = x_1$ (which represent a vertical line).

If $x_1 \neq x_2$, the line through the two points is non-vertical. An equation of a (non-vertical) line through the point $(x_1, y_1)$ is

$$y - y_1 = m(x - x_1),$$

where $m$ is some real number. This line contains the point $(x_2, y_2)$ if and only if

$$y_2 - y_1 = m(x_2 - x_1) \Longleftrightarrow m = \frac{y_2 - y_1}{x_2 - x_1}.$$

So we obtain the equation

$$y - y_1 = \frac{y_2 - y_1}{x_2 - x_1}(x - x_1).$$

(d) The slope of a line perpendicular to a line with slope $m = 2$, is $m = -\frac{1}{2}$. So we get the equation $y = -\frac{1}{2}x$.

1.38 Observe that

$$\frac{ax}{4(x+1)} \Big/ \frac{ax^2 + 4ax}{x^2 - 1} = \frac{ax}{4(x+1)} \Big/ \frac{ax(x+4)}{(x-1)(x+1)}$$

this expression is not defined for $x = -4, x = -1, x = 0$ and $x = 1$.

Further, for $x \notin \{-4, -1, 0, 1\}$,

$$\frac{ax}{4(x+1)} \Big/ \frac{ax^2 + 4ax}{x^2 - 1} = \frac{ax}{4(x+1)} \Big/ \frac{ax(x+4)}{(x-1)(x+1)} = \frac{ax}{4(x+1)} \cdot \frac{(x-1)(x+1)}{ax(x+4)} = \frac{x-1}{4(x+4)}.$$

1.40 We have

$${}^2\log 3 + {}^4\log 3 = {}^2\log 3 + \frac{{}^2\log 3}{{}^2\log 4} = {}^2\log 3 + \frac{{}^2\log 3}{2} = 1\tfrac{1}{2}\,{}^2\log 3.$$