1.9 According to Theorem 1,

$$\sum_{i=2}^{n}\frac{1}{i(i-1)}=\sum_{i=2}^{n}\left[\frac{1}{i-1}-\frac{1}{i}\right]=\sum_{i=2}^{n}\frac{1}{i-1}-\sum_{i=2}^{n}\frac{1}{i}=\sum_{j=1}^{n-1}\frac{1}{j}-\sum_{i=2}^{n}\frac{1}{i}=1-\frac{1}{n}.$$

2.2 (a) For $n\in\mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{n(n+1)}=\dfrac{n}{n+1}$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $\dfrac{1}{1(1+1)}=\frac{1}{2}=\dfrac{1}{1+1}$.

(2) Let $k\in\mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $\dfrac{1}{1\cdot 2}+\dfrac{1}{2\cdot 3}+\dfrac{1}{3\cdot 4}+\cdots+\dfrac{1}{k(k+1)}=\dfrac{k}{k+1}$.

Then

$$\begin{aligned}\frac{1}{1\cdot 2}+\frac{1}{2\cdot 3}+\frac{1}{3\cdot 4}+\cdots+\frac{1}{k(k+1)}+\frac{1}{(k+1)(k+2)}&=\frac{k}{k+1}+\frac{1}{(k+1)(k+2)}\\&=\frac{k^2+2k+1}{(k+1)(k+2)}=\frac{(k+1)^2}{(k+1)(k+2)}\\&=\frac{k+1}{k+2}=\frac{k+1}{(k+1)+1}.\end{aligned}$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n\in\mathbb{N}$.

(b) For $n\in\mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $n<2^n$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $1<2=2^1$.

(2) Let $k\in\mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $k<2^k$.

Then

$$k+1<2^k+1<2^k+2^k=2\cdot 2^k=2^{k+1}.$$

(alternatively : $\quad k+1\le k+k=2k<2\cdot 2^k=2^{k+1}$.)

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n\in\mathbb{N}$.

2.3 Let $r\in\mathbb{R}\setminus\{1\}$

For $n\in\mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $1+r+r^2+\cdots+r^n=\dfrac{1-r^{n+1}}{1-r}$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $1+r=\dfrac{1-r^2}{1-r}$.

(2) Let $k\in\mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $1+r+r^2+\cdots+r^k=\dfrac{1-r^{k+1}}{1-r}$.

Then

$$\begin{aligned}1+r+r^2+\cdots+r^k+r^{k+1}&=\frac{1-r^{k+1}}{1-r}+r^{k+1}\\&=\frac{1-r^{k+1}}{1-r}+\frac{r^{k+1}-r^{k+2}}{1-r}\\&=\frac{1-r^{k+2}}{1-r}.\end{aligned}$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n\in\mathbb{N}$.

2.4 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{n}} > \sqrt{n}$.

(1) Since we have to prove that the statement $\mathcal{P}(n)$ is true for $n \geq 2$, we have to check in this case whether the statement $\mathcal{P}(2)$ is true!

Indeed, $\mathcal{P}(2)$ is true because $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} > \sqrt{2}$ (check it yourself).

(2) Let $k \in \mathbb{N}$, $k \geq 2$ and assume that $\mathcal{P}(k)$ is true, that is: $\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} > \sqrt{k}$.

Then

$$\begin{aligned}\frac{1}{\sqrt{1}} + \frac{1}{\sqrt{2}} + \cdots + \frac{1}{\sqrt{k}} + \frac{1}{\sqrt{k+1}} &> \sqrt{k} + \frac{1}{\sqrt{k+1}} = \frac{\sqrt{k}\sqrt{k+1}+1}{\sqrt{k+1}} \\ &= \frac{\sqrt{k^2+k}+1}{\sqrt{k+1}} > \frac{\sqrt{k^2}+1}{\sqrt{k+1}} = \frac{k+1}{\sqrt{k+1}} = \sqrt{k+1}.\end{aligned}$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \geq 2$.

2.5 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $2^n \leq (n+1)!$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $2^1 = 2 \leq 2 = 2! = (1+1)!$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $2^k \leq (k+1)!$.

Then

$$2^{k+1} = 2 \cdot 2^k \leq 2(k+1)! \leq (k+2)(k+1)! = (k+2)! = \big((k+1)+1\big)!.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

2.6 (a) By using the formula for $\binom{n}{r}$ we find that

$$\begin{aligned}\binom{n}{0} &= \frac{n!}{(n-0)!\,0!} = \frac{n!}{n!} = 1, \\ \binom{n}{1} &= \frac{n!}{(n-1)!\,1!} = \frac{n!}{(n-1)!} = \frac{n(n-1)!}{(n-1)!} = n, \\ \binom{n}{n-1} &= \frac{n!}{1!(n-1)!} = \frac{n!}{(n-1)!} = n\end{aligned}$$

and

$$\binom{n}{n} = \frac{n!}{0!\,n!} = \frac{n!}{n!} = 1.$$

(b) Let $n \in \mathbb{N}$ and $i \in \{1, 2, \ldots, n\}$. Then

$$\begin{aligned}\binom{n}{i-1} + \binom{n}{i} &= \frac{n!}{(n-i+1)!\,(i-1)!} + \frac{n!}{(n-i)!\,i!} \\ &= \frac{n!}{(n-i+1)(n-i)!\,(i-1)!} + \frac{n!}{(n-i)!\,i(i-1)!} \\ &= \frac{in!}{(n-i+1)(n-i)!\,i(i-1)!} + \frac{(n-i+1)n!}{(n-i+1)(n-i)!\,i(i-1)!} \\ &= \frac{n!\,(i+n-i+1)}{(n-i+1)!\,i!} = \frac{n!\,(n+1)}{(n-i+1)!\,i!} \\ &= \frac{(n+1)!}{(n+1-i)!\,i!} = \binom{n+1}{i}.\end{aligned}$$

2.7 (a) If we choose $a = 1$ and $b = 1$, the Binomial Formula leads to

$$2^n = (1+1)^n = \sum_{i=0}^{n} \binom{n}{i} 1^{n-i} 1^i = \sum_{i=0}^{n} \binom{n}{i}.$$

(b) If we choose $a = 1$ and $b = 1$, the Binomial Formula leads to

$$2^{n-1} = \sum_{i=0}^{n-1} \binom{n-1}{i} = \underbrace{\binom{n-1}{0} + \binom{n-1}{1}}_{=\binom{n}{1}} + \underbrace{\binom{n-1}{2} + \binom{n-1}{3}}_{=\binom{n}{3}} + \cdots + \binom{n-1}{n-1}$$

$$= \binom{n}{1} + \binom{n}{3} + \cdots.$$

Here we used Exercise 6 (b).

2.8 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $\prod_{i=1}^{n}(1+\frac{2}{i}) < (n+1)^2$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $1 + \frac{2}{1} = 3 < 4 = (1+1)^2$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $\prod_{i=1}^{k}(1+\frac{2}{i}) < (k+1)^2$.

Then

$$\prod_{i=1}^{k+1}(1+\frac{2}{i}) = \Big[\prod_{i=1}^{k}(1+\frac{2}{i})\Big] \cdot (1+\frac{2}{k+1}) < (k+1)^2(1+\frac{2}{k+1})$$
$$= (k+1)^2 + 2(k+1) = k^2 + 4k + 3 < k^2 + 4k + 4 = (k+2)^2.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.