2.1 (a) For $n \in \mathbb{N}$ we introduce the statement $\mathcal{Q}(n)$: $9^n - 5$ is not a multiple of 8.

(1) First we show that the statement $\mathcal{Q}(1)$ is true: $9^1 - 5 = 4$ is not a multiple of 8.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{Q}(k)$ is true, that is: $9^k - 5$ is not a multiple of 8.

Suppose that $9^{k+1} - 5$ is a multiple of 8. Then

$$9^{k+1} - 5 = 9 \cdot 9^k - 5 = 8 \cdot 9^k + 9^k - 5 \Longrightarrow 9^k - 5 = \underbrace{9^{k+1} - 5}_{\text{is a multiple of 8}} - \underbrace{8 \cdot 9^k}_{\text{is a multiple of 8}},$$

which is a contradiction.

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{Q}(n)$ is true for all $n \in \mathbb{N}$.

(b) Note that

$$9^n - 5 = 9^n - 1 - 4 = (9-1)(9^{n-1} + 9^{n-2} + \cdots + 1) - 4 = 8(9^{n-1} + 9^{n-2} + \cdots + 1) - 4$$

$$= \underbrace{4 \cdot \left[2(9^{n-1} + 9^{n-2} + \cdots + 1) - 1\right]}_{\text{is a multiple of 4}}.$$

2.9 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $\sum_{i=1}^{n} i^2 = \frac{1}{6}n(n+1)(2n+1)$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $1^2 = 1 = \frac{1}{6} \cdot 1 \cdot 2 \cdot 3$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $\sum_{i=1}^{k} i^2 = \frac{1}{6}k(k+1)(2k+1)$.

Then

$$\sum_{i=1}^{k+1} i^2 = \sum_{i=1}^{k} i^2 + (k+1)^2 = \frac{1}{6}k(k+1)(2k+1) + (k+1)^2$$

$$= \frac{1}{6}(k+1)\left[k(2k+1) + 6(k+1)\right] = \frac{1}{6}(k+1)\left[2k^2 + 7k + 6\right] = \frac{1}{6}(k+1)\left[(2k+3)(k+2)\right]$$

$$= \frac{1}{6}(k+1)(k+2)(2k+3) = \frac{1}{6}(k+1)[(k+1)+1][2(k+1)+1].$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

2.10 Let $x \in (-1,1)$. Since the relation holds for $x = 0$, we will assume that $x \neq 0$. Then $0 < |x| < 1$.

For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $|x^n| \leq |x|$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $|x^1| \leq |x|$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $|x^k| \leq |x|$.

Then, according to Theorem 1.2 (b),

$$|x^{k+1}| = |x^k \cdot x| = |x^k| \cdot |x| \underset{|x|<1}{\leq} |x^k| \leq |x|.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$. Since x was arbitrarily chosen, the statement is true for all $x \in (-1,1)$ and all $n \in \mathbb{N}$.

2.11 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $\sum_{i=1}^{n} \frac{1}{i!} \leq 2 - \frac{1}{2^{n-1}}$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $1 \leq 2 - 1$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $\sum_{i=1}^{k} \frac{1}{i!} \leq 2 - \frac{1}{2^{k-1}}$.

Then

$$\sum_{i=1}^{k+1} \frac{1}{i!} = \sum_{i=1}^{k} \frac{1}{i!} + \frac{1}{(k+1)!} \leq 2 - \frac{1}{2^{k-1}} + \frac{1}{(k+1)!}$$

$$\underset{\text{Exercise 5}}{\leq} 2 - \frac{1}{2^{k-1}} + \frac{1}{2^k} = 2 - 2 \cdot \frac{1}{2^k} + \frac{1}{2^k} = 2 - \frac{1}{2^k}.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

2.12 Let $n \in \mathbb{N}$ and let $k \in \{1, 2, \cdots, n\}$. Then

$$k\binom{n}{k} = \frac{k \cdot n!}{(n-k)!\,k!} = \frac{n!}{(n-k)!\,(k-1)!}$$

and

$$n\binom{n-1}{k-1} = \frac{n(n-1)!}{(n-k)!\,(k-1)!} = \frac{n!}{(n-k)!\,(k-1)!}.$$

2.13 We will determine the sum of all the numbers in the array by first finding the sum of the numbers in each of the m rows. The sum of the elements in the ith row is

$$a_{i1} + a_{i2} + \cdots + a_{ii} = \sum_{j=1}^{i} a_{ij}.$$

Adding all these row sums leads to

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{i} a_{ij} \right).$$

Next we will determine the sum of all the numbers in the array by first finding the sum of the numbers in each of the n columns. The sum of the elements in the jth column is

$$a_{jj} + a_{j+1j} + \cdots + a_{mj} = \sum_{i=j}^{m} a_{ij}.$$

Adding all these column sums leads to

$$\sum_{j=1}^{m} \left(\sum_{i=j}^{m} a_{ij} \right).$$

Obviously, the two results are equal:

$$\sum_{i=1}^{m} \left(\sum_{j=1}^{i} a_{ij} \right) = \sum_{j=1}^{m} \left(\sum_{i=j}^{m} a_{ij} \right).$$

2.14 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $t_n = 2^n$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $t_1 = 2^1 = 2$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $t_k = 2^k$.

Then

$$t_{k+1} = 2t_k = 2 \times 2^k = 2^{k+1}.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

2.15 The numbers $a-1$ and $b-1$, which play a role in part (b) of the 'proof', can be equal to zero. As zero is not a natural number, you cannot apply the induction hypothesis.

2.16 Let $0 < a < b$. For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $a^n < b^n$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $a < b$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $a^k < b^k$.

Then

$$a^{k+1} = a \cdot a^k < a \cdot b^k < b \cdot b^k = b^{k+1}.$$

[Note that the first inequality is obtained by multiplying both sides of the inequality $a^k < b^k$ by the positive number a. The second inequality is obtained by multiplying both sides of the inequality $a < b$ by the positive number b^k.]

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.