3.4 My pocket calculator can represent only 8 digits. So I try to figure out when the terms of the sequences are equal to 0.6666666.

If $n = 6 \times 10^6$, then $t_n = 0.6666665$.

If $n = 7 \times 10^6$, then $t_n = 0.6666666$.

Note that $v_n = t_{n^2}$.

If $n = 2 \times 10^3$, then $v_n = 0.6666665$.

If $n = 3 \times 10^3$, then $v_n = 0.6666666$.

If n is odd, $w_n = t_n$.

If $n = 6 \times 10^6$, then $w_n = 0.6666665$.

If $n = 7 \times 10^6$, then $w_n = 0.6666667$.

3.5 Let, for $n \in \mathbb{N}$, $s_n = (-1)^n$ and $t_n = (-1)^{n+1}$.

Since

$$s_n + t_n = (-1)^n + (-1)^{n+1} = (-1)^n - (-1)^n = 0 \text{ for all } n \in \mathbb{N},$$

the sum of the sequences $\big(s_n\big)_{n=1}^{\infty}$ and $\big(t_n\big)_{n=1}^{\infty}$ is the sequence $0, 0, 0, 0, \ldots$.

Since

$$s_n - t_n = (-1)^n - (-1)^{n+1} = (-1)^n + (-1)^n = 2(-1)^n \text{ for all } n \in \mathbb{N},$$

the difference of the sequences $\big(s_n\big)_{n=1}^{\infty}$ and $\big(t_n\big)_{n=1}^{\infty}$ is the sequence $\big(2(-1)^n\big)_{n=1}^{\infty}$.

Since

$$s_n \cdot t_n = (-1)^n \cdot (-1)^{n+1} = (-1)^{2n+1} = -1 \text{ for all } n \in \mathbb{N},$$

the product of the sequences $\big(s_n\big)_{n=1}^{\infty}$ and $\big(t_n\big)_{n=1}^{\infty}$ is the sequence $-1, -1, -1, \ldots$.

Since

$$\frac{s_n}{t_n} = \frac{(-1)^n}{(-1)^{n+1}} = (-1)^{-1} = -1 \text{ for all } n \in \mathbb{N},$$

the quotient of the sequences $\big(s_n\big)_{n=1}^{\infty}$ and $\big(t_n\big)_{n=1}^{\infty}$ is the sequence $-1, -1, -1, \ldots$.

3.8 (a) Since

$$\frac{1}{\sqrt{n}} < \tfrac{1}{20} \Longleftrightarrow \sqrt{n} > 20 \Longleftrightarrow n > 400,$$

one can choose $N = 400$.

Since

$$\frac{1}{\sqrt{n}} < \tfrac{1}{101} \Longleftrightarrow \sqrt{n} > 101 \Longleftrightarrow n > 10\,201,$$

one can choose $N = 10\,201$.

Since

$$\frac{1}{\sqrt{n}} < 10^{-4} \Longleftrightarrow \sqrt{n} > 10^4 \Longleftrightarrow n > 10^8,$$

one can choose $N = 10^{-8}$.

(b) Since

$$\frac{1}{\sqrt{n}} < \varepsilon \Longleftrightarrow \sqrt{n} > \frac{1}{\varepsilon} \Longleftrightarrow n > \frac{1}{\varepsilon^2},$$

one can choose $N = 1/\varepsilon^2$.

(c) Obviously, $\lim_{n\to\infty} \frac{1}{\sqrt{n}} = 0$. Indeed, if $\varepsilon > 0$, then, according to part (b),

$$\left|\frac{1}{\sqrt{n}} - 0\right| < \varepsilon,$$

whenever $n > \frac{1}{\varepsilon^2}$.

3.9 (a) We suspect that the limit of the sequence $\left(\frac{1}{n+7}\right)_{n=1}^{\infty}$ is 0. For $\varepsilon = 0.1$ we have

$$\left|\frac{1}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \tfrac{1}{10} \iff n+7 > 10 \iff n > 3 = N_{0.1}$$

For $\varepsilon = 0.01$ we have

$$\left|\frac{1}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \tfrac{1}{100} \iff n+7 > 100 \iff n > 93 = N_{0.01}.$$

For an arbitrary ε, we have

$$\left|\frac{1}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \varepsilon \iff n+7 > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon} - 7 = N_{\varepsilon}.$$

(b) We suspect that the limit of the sequence $\left(\frac{n}{n+7}\right)_{n=1}^{\infty}$ is 1. For $\varepsilon = 0.1$ we have

$$\left|\frac{n}{n+7} - 1\right| < \varepsilon \iff \frac{7}{n+7} < \tfrac{1}{10} \iff n+7 > 70 \iff n > 63 = N_{0.1}$$

For $\varepsilon = 0.01$ we have

$$\left|\frac{n}{n+7} - 1\right| < \varepsilon \iff \frac{7}{n+7} < \tfrac{1}{100} \iff n+7 > 700 \iff n > 693 = N_{0.01}.$$

For an arbitrary ε, we have

$$\left|\frac{n}{n+7} - 1\right| < \varepsilon \iff \frac{7}{n+7} < \varepsilon \iff n+7 > \frac{7}{\varepsilon} \iff n > \frac{7}{\varepsilon} - 7 = N_{\varepsilon}.$$

(c) We suspect that the limit of the sequence $\left(\frac{(-1)^{n+1}}{n+7}\right)_{n=1}^{\infty}$ is 0. For $\varepsilon = 0.1$ we have

$$\left|\frac{(-1)^{n+1}}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \tfrac{1}{10} \iff n+7 > 10 \iff n > 3 = N_{0.1}$$

For $\varepsilon = 0.01$ we have

$$\left|\frac{(-1)^{n+1}}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \tfrac{1}{100} \iff n+7 > 100 \iff n > 93 = N_{0.01}.$$

For an arbitrary ε, we have

$$\left|\frac{(-1)^{n+1}}{n+7} - 0\right| < \varepsilon \iff \frac{1}{n+7} < \varepsilon \iff n+7 > \frac{1}{\varepsilon} \iff n > \frac{1}{\varepsilon} - 7 = N_{\varepsilon}.$$

3.10 (a) Let $\varepsilon > 0$. Note that

$$\left|\frac{1}{n} - 0\right| < \varepsilon \iff \frac{1}{n} < \varepsilon \iff n > \frac{1}{\varepsilon}.$$

So if we choose $N = \frac{1}{\varepsilon}$, then for all $n > N$,

$$\left|\frac{1}{n} - 0\right| = \frac{1}{n} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{1}{n} = 0$.

3.11 Note that

$$\left|\frac{1}{3n} - 0\right| < \varepsilon \iff \frac{1}{3n} < \varepsilon \iff n > \frac{1}{3\varepsilon}$$

and

$$\left|\frac{3}{n} - 0\right| < \varepsilon \iff n > \frac{3}{\varepsilon}.$$

Therefore we choose $N = \max\{\frac{1}{3\varepsilon}, \frac{3}{\varepsilon}\} = \frac{3}{\varepsilon}$.

Then, for all $n > N$,

$$\text{if } n \text{ is even} \quad |a_n - 0| = \frac{1}{3n} < \frac{\varepsilon}{9} < \varepsilon$$

$$\text{if } n \text{ is odd} \quad |a_n - 0| = \frac{3}{n} < \varepsilon$$

This proves that $\lim_{n\to\infty} a_n = 0$.

3.12 (a) Let $\varepsilon > 0$. Note that

$$\left|\frac{2n-1}{n+2} - 2\right| = \left|\frac{2n-1}{n+2} - \frac{2n+4}{n+2}\right| = \left|\frac{-5}{n+2}\right| = \frac{5}{n+2}.$$

Since

$$\frac{5}{n+2} < \varepsilon \iff n+2 > \frac{5}{\varepsilon} \iff n > \frac{5}{\varepsilon} - 2,$$

we choose $N = \frac{5}{\varepsilon} - 2$. Then, for all $n > N$,

$$\left|\frac{2n-1}{n+2} - 2\right| = \frac{5}{n+2} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{2n-1}{n+2} = 2$.

(b) Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$\left|\frac{1-n^2}{2n^2+1} + \tfrac{1}{2}\right| = \left|\frac{1-n^2+n^2+\frac{1}{2}}{2n^2+1}\right| = \frac{\frac{3}{2}}{2n^2+1} < \frac{4}{2n^2}.$$

Since

$$\frac{4}{2n^2} < \varepsilon \iff n^2 > \frac{2}{\varepsilon} \iff n > \sqrt{\frac{2}{\varepsilon}},$$

we choose $N = \sqrt{\frac{2}{\varepsilon}}$.

Then, for all $n > N$,

$$\left|\frac{1-n^2}{2n^2+1} + \tfrac{1}{2}\right| = \frac{\frac{3}{2}}{2n^2+1} < \frac{2}{n^2} < \frac{2}{\frac{2}{\varepsilon}} = \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{1-n^2}{2n^2+1} = -\frac{1}{2}$.

3.15 Assume that the sequence $\left(t_n\right)_{n=1}^{\infty}$ converges in our sense.

Let $k \in \mathbb{N}$. If we choose $\varepsilon = 10^{-k}$, then a number $N \in \mathbb{R}$ can be found such that

$$|t_n - \ell| < \varepsilon = 10^{-k},$$

whenever $n > N$. So the sequence converges in the other sense.

Next assume that the sequence $\left(t_n\right)_{n=1}^{\infty}$ converges in the sense described in the exercise.

Let $\varepsilon > 0$. Choose a natural number k such that $10^{-k} < \varepsilon$.

Then a number $N \in \mathbb{R}$ can be found such that

$$|t_n - \ell| < 10^{-k} < \varepsilon,$$

whenever $n > N$. So the sequence converges in the our sense.