3.6 (a) We determine the first coordinate of the intersection S of the line segment joining P and Q with the line $y = 2$ (as indicated in Figure 3).

Since

$$y = (2 + t_k)x - 2t_k$$

is the line through the points P and Q, the term t_{k+1} satisfies

$$(2 + t_k)t_{k+1} - 2t_k = 2 \Longrightarrow t_{k+1} = \frac{2 + 2t_k}{2 + t_k}.$$

So the sequence $\left(t_n\right)_{n=1}^{\infty}$ is (recursively) defined by

$$t_1 = \tfrac{1}{2}$$
$$t_{n+1} = \frac{2 + 2t_n}{2 + t_n} \quad \text{for } n \in \mathbb{N}.$$

(b) We will prove by induction that $t_n > 0$ for all $n \in \mathbb{N}$.

(1) Clearly $t_1 = \frac{1}{2} > 0$.

(2) Let $k \in \mathbb{N}$ and assume that $t_k > 0$. Then

$$t_{k+1} = \frac{\overbrace{2 + 2t_k}^{>0}}{\underbrace{2 + t_k}_{>0}} > 0.$$

According to the Principle of Induction, $t_n > 0$ for all $n \in \mathbb{N}$.

3.7 For $k = 1$, $N = 3$. For $k = 2$, $N = 5$. For $k = 3$, $N = 6$. For $k = 4$, $N = 8$. For $k = 5$, $N = 10$. For $k = 6$, $N = 11$. For $k = 7$, $N = 13$ For $k = 8$, $N = 15$.

3.9 (d) We suspect that the limit of the sequence $\left(\frac{1}{n!}\right)_{n=1}^{\infty}$ is 0. For $\varepsilon = 0.1$ we have

$$\left|\frac{1}{n!} - 0\right| < \varepsilon \Longleftrightarrow \frac{1}{n!} < \tfrac{1}{10} \Longleftrightarrow n! > 10 \Longleftrightarrow n > 4.$$

For $\varepsilon = 0.01$ we have

$$\left|\frac{1}{n!} - 0\right| < \varepsilon \Longleftrightarrow \frac{1}{n!} < \tfrac{1}{100} \Longleftrightarrow n! > 100 \Longleftrightarrow n > 5.$$

For an arbitrary ε, we have

$$\left|\frac{1}{n!} - 0\right| < \varepsilon \Longleftrightarrow \frac{1}{n!} < \varepsilon \Longleftrightarrow n! > \frac{1}{\varepsilon}.$$

As $n! \geq n$, we choose $n > \frac{1}{\varepsilon}$. Then for $n > \varepsilon^{-1}$,

$$n! \geq n > \frac{1}{\varepsilon} \Longrightarrow \left|\frac{1}{n!} - 0\right| < \varepsilon.$$

(e) We suspect that the limit of the sequence $\left(\frac{2n}{n^3 + 7}\right)_{n=1}^{\infty}$ is 0. For $\varepsilon = 0.1$ we have

$$\left|\frac{2n}{n^3 + 7} - 0\right| < \varepsilon \Longleftrightarrow \frac{2n}{n^3 + 7} < \tfrac{1}{10} \Longleftrightarrow n^3 + 7 > 20n \Longleftrightarrow n^3 - 20n + 7 > 0$$
$$\Longleftarrow n^2 - 20n + 7 > 0 \Longleftrightarrow n > \frac{20 + \sqrt{400 - 28}}{2}.$$

So we may choose $N = 20$.

For $\varepsilon = 0.01$ we have

$$\left|\frac{2n}{n^3+7} - 0\right| < \varepsilon \iff \frac{2n}{n^3+7} < \tfrac{1}{100} \iff n^3 + 7 > 200n \iff n^3 - 200n + 7 > 0$$
$$\Longleftarrow n^2 - 200n + 7 > 0 \iff n > \frac{200 + \sqrt{40000-28}}{2}.$$

So we may choose $N = 200$.

For an arbitrary ε, we have

$$\left|\frac{2n}{n^3+7} - 0\right| < \varepsilon \iff \frac{2n}{n^3+7} < \varepsilon \iff n^3 + 7 > \frac{2n}{\varepsilon} \iff n^3 - \frac{2n}{\varepsilon} + 7 > 0$$
$$\Longleftarrow n^2 - \frac{2n}{\varepsilon} + 7 > 0 \iff n > \frac{\frac{2}{\varepsilon} + \sqrt{\frac{4}{\varepsilon^2} - 28}}{2} = \frac{1}{\varepsilon} + \sqrt{\frac{1-7\varepsilon^2}{\varepsilon^2}} = \frac{1}{\varepsilon} + \frac{1}{\varepsilon}\sqrt{1-7\varepsilon^2}$$

So we may choose $N = \frac{2}{\varepsilon}$.

(f) We suspect that the limit of the sequence $\left(\frac{n^2}{n^2+7}\right)_{n=1}^{\infty}$ is 1.

Observe that

$$\left|\frac{n^2}{n^2+7} - 1\right| < \varepsilon \iff \left|\frac{n^2-n^2-7}{n^2+7}\right| < \varepsilon \iff \frac{7}{n^2+7} < \varepsilon \Longleftarrow \frac{7}{n^2} < \varepsilon \iff n^2 > \frac{7}{\varepsilon} \iff n > \sqrt{\frac{7}{\varepsilon}}.$$

So we may choose $N = \sqrt{\frac{7}{\varepsilon}}$.

By choosing ε equal to the value $0.1, 0.01 \ldots$, we can find the proper 'big N'.

3.10 (b) Let $\varepsilon > 0$. Note that for all $n \in \mathbb{N}$

$$\left|1 + \frac{(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n}.$$

Since $\frac{1}{n} < \varepsilon$, whenever $n > \frac{1}{\varepsilon}$, we choose $N = \frac{1}{\varepsilon}$.

Then, for all $n > N$,

$$\left|1 + \frac{(-1)^n}{n} - 1\right| = \left|\frac{(-1)^n}{n}\right| = \frac{1}{n} < \frac{1}{\frac{1}{\varepsilon}} = \varepsilon.$$

This proves that $\lim_{n\to\infty} 1 + \frac{(-1)^n}{n} = 1$.

3.12 (c) Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$\left|\sqrt{\frac{n+1}{n^2}} - 0\right| = \sqrt{\frac{1}{n^2}(n+1)} = \sqrt{\frac{1}{n} + \frac{1}{n^2}} \leq \sqrt{\frac{1}{n} + \frac{1}{n}} < \sqrt{\frac{2}{n}}.$$

As $\sqrt{\frac{2}{n}} < \varepsilon \iff \frac{2}{n} < \varepsilon^2 \iff n > \frac{2}{\varepsilon^2}$, we choose $N = \frac{2}{\varepsilon^2}$.

Then, for all $n > N$,

$$\sqrt{\frac{n+1}{n^2}} < \sqrt{\frac{2}{n}} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \sqrt{\frac{n+1}{n^2}} = 0$.

3.13 Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$(\tfrac{2}{3})^n = \frac{1}{(\frac{3}{2})^n} = \frac{1}{(1+\frac{1}{2})^n} \leq \frac{1}{1+\frac{1}{2}n} < \frac{1}{\frac{1}{2}n} = \frac{2}{n}.$$

Here the first inequality is based on Bernoulli's Inequality:

$$\left(1+\tfrac{1}{2}\right)^n \geq 1 + \tfrac{1}{2}n.$$

Since $\dfrac{2}{n} < \varepsilon$ whenever $n > \dfrac{2}{\varepsilon}$, we choose $N = \dfrac{2}{\varepsilon}$.
Then, for all $n > N$,

$$\left|(\tfrac{2}{3})^n - 0\right| = (\tfrac{2}{3})^n = \frac{1}{(\frac{3}{2})^n} = \frac{1}{(1+\frac{1}{2})^n} \leq \frac{1}{1+\frac{1}{2}n} < \frac{1}{\frac{1}{2}n} = \frac{2}{n} < \frac{2}{\frac{2}{\varepsilon}} = \varepsilon.$$

This proves that $\lim_{n\to\infty} (\frac{2}{3})^n = 0$.

3.14 Let $\varepsilon > 0$. As $\lim_{n\to\infty} t_n = \ell$, a number N exists such that

$$|t_n - \ell| < \varepsilon,$$

whenever $n > N$.

As for $n > N$,

$$|t_n - \ell| < \varepsilon \Longleftrightarrow -\varepsilon < t_n - \ell < \varepsilon \Longleftrightarrow -\varepsilon < a_n < \varepsilon \Longleftrightarrow |a_n| < \varepsilon,$$

it follows that $\lim_{n\to\infty} a_n = 0$.

As for $n > N$,

$$|t_n - \ell| < \varepsilon \Longleftrightarrow -\varepsilon < t_n - \ell < \varepsilon \Longleftrightarrow 2\ell - \varepsilon < t_n + \ell < 2\ell\varepsilon \Longleftrightarrow 2\ell - \varepsilon < b_n < 2\ell + \varepsilon$$

$$\Longleftrightarrow -\varepsilon < b_n - 2\ell < \varepsilon \Longleftrightarrow |b_n - 2\ell| < \varepsilon,$$

it follows that $\lim_{n\to\infty} b_n = 2\ell$.