3.18 The constant sequence $1, 1, 1, \dots$ is a sequence with positive terms and 1 as the positive limit.

The sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ is a sequence with positive terms and 0 as its limit.

3.19 (a) We give a prove by contradiction.

Suppose that $\ell > b$. Choose $\varepsilon = \ell - b$. Then for this ε an N exists such that, for all $n > N$,

$$|t_n - \ell| < \varepsilon = \ell - b \Longleftrightarrow -(\ell - b) < t_n - \ell < \ell - b \Longleftrightarrow b < t_n < 2\ell - b.$$

The first inequality leads to a contradiction.

(b) If $I = [a, b]$, for some $a, b \in \mathbb{R}$, then $t_n \geq a$ for all $n \in \mathbb{N}$. Hence, according to an obvious version of part (a), $\ell \geq a$. Similarly, $\ell \leq b$, that is: $\ell \in I$.

The cases $I = [a, \infty)$ and $I = (-\infty, b]$, where $a, b \in \mathbb{R}$ are similar.

3.20 The proof consists of two parts.

(a) If an $m > 0$ exists such that

$$|t_n| \leq m,$$

for all $n \in \mathbb{N}$, then $-m \leq t_n \leq m$ for all $n \in \mathbb{N}$. Hence, $-m$ is a lower bound of the sequence and m is an upper bound.

(b) Suppose that the sequence $\left(t_n\right)_{n=1}^{\infty}$ is bounded. Then numbers l and u exist such that

$$l \leq t_n \leq u,$$

for al $n \in \mathbb{N}$. According to Exercise 1.11, this implies that for al $n \in \mathbb{N}$,

$$-|l| \leq l \leq t_n \leq u \leq |u|.$$

Now choose $m = \max\{|l|, |u|\}$. Since $|u| \leq \max\{|l|, |u|\} = m$ and $-|l| \geq -\max\{|l|, |u|\} = -m$, we obtain that, for all $n \in \mathbb{N}$,

$$-m \leq -|l| \leq t_n \leq |u| \leq m \Longleftrightarrow |t_n| \leq m.$$

3.22 Assume that $\lim_{n\to\infty} t_n = 0$.

Let $u \in \mathbb{R}$. We will show that a k exists such that

$$\frac{1}{t_k} > u.$$

We may suppose w.l.o.g. (without loss of generality) that $u > 0$.

Because $\lim_{n\to\infty} t_n = 0$, for $\varepsilon = \frac{1}{u}$ a number N exists such that

$$|t_n| < \varepsilon = \frac{1}{u} \Longrightarrow t_n < \varepsilon = \frac{1}{u},$$

whenever $n > N$.

Then for any $k > N$, $\frac{1}{t_k} > u$. Hence, the sequence $\left(t_n\right)_{n=1}^{\infty}$ is unbounded.

3.25 (b) Note that, for all $n \in \mathbb{N}$,

$$\frac{1-n^2}{2n^2+1} = \frac{\frac{1}{n^2} - 1}{2 + \frac{1}{n^2}}.$$

Because the sequence $\left(\frac{1}{n}\right)_{n=1}^{\infty}$ converges to 0, Theorem 3 (b) implies that $\frac{1}{n^2} \to 0$ as $n \to \infty$. Then, according to Theorem 3 (a), $\frac{1}{n^2} - 1 \to -1$ as $n \to \infty$ and $2 + \frac{1}{n^2} \to 2$ as $n \to \infty$.

Finally, Theorem 3 (c) implies that

$$\frac{1-n^2}{2n^2+1} = \frac{\frac{1}{n^2} - 1}{2 + \frac{1}{n^2}} \to \frac{-1}{2} = -\tfrac{1}{2} \quad \text{as} \quad n \to \infty.$$

3.29 (a) We will prove that $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$.

Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$\left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} \leq \frac{2}{n^2}.$$

Since $\frac{2}{n^2} < \varepsilon \Longleftrightarrow n > \sqrt{\frac{2}{\varepsilon}}$, we choose $N = \sqrt{\frac{2}{\varepsilon}}$. Then, for all $n > N$,

$$\left|\frac{n^2-1}{n^2+1} - 1\right| = \frac{2}{n^2+1} \leq \frac{2}{n^2} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{n^2-1}{n^2+1} = 1$.

(b) We will prove that $\lim_{n\to\infty} \frac{\sqrt{n}+1}{n} = 0$.

Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$\left|\frac{\sqrt{n}+1}{n}\right| = \frac{1}{\sqrt{n}} + \frac{1}{n} \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} = \frac{2}{\sqrt{n}}.$$

Since $\frac{2}{\sqrt{n}} < \varepsilon \Longleftrightarrow n > \frac{4}{\varepsilon^2}$, we choose $N = \frac{4}{\varepsilon^2}$. Then, for all $n > N$,

$$\left|\frac{\sqrt{n}+1}{n}\right| = \frac{1}{\sqrt{n}} + \frac{1}{n} \leq \frac{2}{\sqrt{n}} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{\sqrt{n}+1}{n} = 0$.

3.30 Note that, for all $n \in \mathbb{N}$,

$$|y_n - \ell| = \left|x_n + \frac{x_n}{n} - \ell\right| \leq |x_n - \ell| + \frac{|x_n|}{n}.$$

Since $\left(x_n\right)_{n=1}^{\infty}$ is a convergent sequence, the sequence is bounded. Hence, there exists a $u > 0$ such that $|x_n| \leq u$ for all $n \in \mathbb{N}$.

So, for all $n \in \mathbb{N}$,

$$|y_n - \ell| = \left|x_n + \frac{x_n}{n} - \ell\right| \leq |x_n - \ell| + \frac{|x_n|}{n} \leq |x_n - \ell| + \frac{u}{n}.$$

Because $x_n \to \ell$ as $n \to \infty$, there exists an N_1 such that

$$|x_n - \ell| < \tfrac{1}{2}\varepsilon,$$

whenever $n > N_1$.

Because $\frac{u}{n} \to 0$ as $n \to \infty$, there exists an N_2 such that

$$\frac{u}{n} < \tfrac{1}{2}\varepsilon,$$

whenever $n > N_2$.

Next we choose $N = \max\{N_1, N_2\}$. Then for all $n > N_1$

$$|y_n - \ell| \le |x_n - \ell| + \frac{u}{n} < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon.$$

3.31 (a) We give a proof by contradiction.

Assume that the sequence $\big(a_n\big)_{n=1}^{\infty}$ converges, say to ℓ and that the sequence $\big(a_n + b_n\big)_{n=1}^{\infty}$ converges, say to m. Then, according to the arithmetic rules for limits of sequences,

$$b_n = (a_n + b_n) - a_n \to m - \ell \quad \text{as} \quad n \to \infty,$$

which contradicts the fact that the sequence $\big(b_n\big)_{n=1}^{\infty}$ is divergent.

(b) The alternating sequence $\big((-1)^n\big)_{n=1}^{\infty}$ is divergent. However,

$$(-1)^n \cdot (-1)^n = 1 \to 1 \quad \text{as} \quad n \to \infty.$$

3.36 Observe that

$$t_n = \frac{n^2 + 7n + \pi}{n^3 + n\pi + \ln 7} = \frac{\dfrac{1}{n} + \dfrac{7}{n^2} + \dfrac{\pi}{n^3}}{1 + \dfrac{\pi}{n^2} + \dfrac{\ln 7}{n^3}}.$$

As The sequences $\big(\frac{1}{n}\big)_{n=1}^{\infty}$, $\big(\frac{1}{n^2}\big)_{n=1}^{\infty}$ and $\big(\frac{1}{n^3}\big)_{n=1}^{\infty}$ converge to zero, the Arithmetic Rules for limits of sequences imply that the sequence $\big(t_n\big)_{n=1}^{\infty}$ converges to $\frac{0}{1} = 0$.

3.39 For $k \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $t_n^k \to \ell^k$ as $n \to \infty$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $t_n \to \ell$ as $n \to \infty$.

(2) Let $m \in \mathbb{N}$ and assume that $\mathcal{P}(m)$ is true, that is: $t_n^m \to \ell^m$ as $n \to \infty$.

Then for all n,

$$t_n^{m+1} = t_n^m \cdot t_n.$$

So according to the Arithmetic Rules for limits of sequences,

$$\lim_{n\to\infty} t_n^{m+1} = \lim_{n\to\infty} t_n^m \cdot \lim_{n\to\infty} t_n = \ell^m \cdot \ell = \ell^{m+1}.$$

This proves that $\mathcal{P}(m+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(k)$ is true for all $k \in \mathbb{N}$.