3.21 (a) Observe that a sequence $\big(t_n\big)_{n=1}^{\infty}$ is **not** bounded below if any real number l is **not** a lower bound of the sequence.

You can prove that a number l is not a lower bound of the sequence by finding a(t least one) term of the sequence, say t_k, which is smaller than l.

Let $l \in \mathbb{R}$. We want to find an n such that $t_n < l \Longleftrightarrow \dfrac{1-n^2}{n} < l$.

Note that, for all $n \in \mathbb{N}$,

$$t_n = \frac{1-n^2}{n} = \frac{1}{n} - n \leq 1 - n.$$

Since $1 - n < l$ if $n > 1 - l$, we choose a $k \in \mathbb{N}$ satisfying $k > 1 - l$. Then for this k

$$t_k = \frac{1-k^2}{k} = \frac{1}{k} - k \leq 1 - k < 1 - (1-l) = l.$$

Alternative

Suppose that the sequence is bounded below, say by l. Then $t_n \geq l$ for all $n \in \mathbb{N}$.

However, if we choose an $n \in \mathbb{N}$ satisfying $n > 1 - l$, then

$$t_n = \frac{1-n^2}{n} = \frac{1}{n} - n \leq 1 - n < l.$$

This is a contradiction. So we may conclude that the sequence is not bounded below.

(b) We prove that the sequence is not bounded above.

Note that, for any natural number n,

$$\frac{n+1}{\sqrt{n}} = \sqrt{n} + \frac{1}{\sqrt{n}} > \sqrt{n}.$$

Now let $u \in \mathbb{R}$. Choose a natural number k such that $k > u$. Then for this k

$$t_{k^2} = \frac{k^2+1}{\sqrt{k^2}} > \sqrt{k^2} = k > u.$$

This proves that u is not an upper bound of the given sequence. As u was arbitrarily chosen the sequence is unbounded. So she is divergent.

3.23 (a) For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $a_n \geq 2 \cdot 3^{n-1}$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $a_1 = 2 \geq 2 \cdot 3^0$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $a_k \geq 2 \cdot 3^{k-1}$.

Then

$$a_{k+1} \geq 3 \cdot a_k \geq 3 \cdot 2 \cdot 3^{k-1} = 2 \cdot 3^k.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

(b) We will prove that the sequence $\big(a_n\big)_{n=1}^{\infty}$ is not bounded (above).

Let $u \in \mathbb{R}$. According to Bernouilli's Inequality, for all n,

$$a_n \geq 2 \cdot 3^{n-1} = \tfrac{2}{3} \cdot 3^n = \tfrac{2}{3}(1+2)^n \geq \tfrac{2}{3}(1+2n) \geq \tfrac{1}{2}(1+2n) = n + \tfrac{1}{2} \geq n.$$

So if we choose a $k \in \mathbb{N}$ satisfying $k > u$, then

$$a_k \geq 2 \cdot 3^{k-1} \geq k > u.$$

Hence the sequence $\left(a_n\right)_{n=1}^{\infty}$ is not bounded above.

3.26 First note that for all n

$$a_n \leq t_n \leq b_n \iff a_n - \ell \leq t_n - \ell \leq b_n - \ell.$$

Now we are going to use that fact that $a_n - \ell$ and $b_n - \ell$ can be made as small as we please by choosing n sufficiently large.

Let $\varepsilon > 0$. Then there exist $N_1, N_2 \in \mathbb{R}$ such that

$$|a_n - \ell| < \varepsilon, \tag{1}$$

whenever $n > N_1$, and

$$|b_n - \ell| < \varepsilon, \tag{2}$$

whenever $n > N_2$.

According to (1), $a_n - \ell > -\varepsilon$, whereas (2) implies that $b_n - \ell < \varepsilon$.

Hence, for all $n > \max\{N_1, N_2\}$,

$$-\varepsilon < a_n - \ell \leq t_n - \ell \leq b_n - \ell < \varepsilon,$$

which implies that $|t_n - \ell| < \varepsilon$. This proves that $\lim\limits_{n\to\infty} t_n = \ell$.

3.27 (a) If $\ell = 0$, then $|\sqrt{t_n} - \sqrt{\ell}| = \sqrt{t_n}$.

Let $\varepsilon > 0$. Because $\lim\limits_{n\to\infty} t_n = 0$, there exists an $N \in \mathbb{R}$ such that

$$\underbrace{|t_n - 0|}_{=t_n} < \varepsilon^2,$$

whenever $n > N$. Then, for all $n > N$,

$$\left|\sqrt{t_n} - 0\right| = \sqrt{t_n} < \sqrt{\varepsilon^2} = \varepsilon.$$

This proves that $\lim\limits_{n\to\infty} \sqrt{t_n} = 0 = \sqrt{\ell}$.

(b) If $\ell > 0$, then (apply the root method)

$$|\sqrt{t_n} - \sqrt{\ell}| = \left|\frac{\left(\sqrt{t_n} - \sqrt{\ell}\right)\left(\sqrt{t_n} + \sqrt{\ell}\right)}{\left(\sqrt{t_n} + \sqrt{\ell}\right)}\right| = \frac{|t_n - \ell|}{\sqrt{t_n} + \sqrt{\ell}} \leq \frac{|t_n - \ell|}{\sqrt{\ell}} = \frac{1}{\sqrt{\ell}}\,|t_n - \ell|\,.$$

Let $\varepsilon > 0$. Because $\lim\limits_{n\to\infty} t_n = \ell$, there exists an $N \in \mathbb{R}$ such that

$$|t_n - \ell| < \varepsilon\sqrt{\ell},$$

whenever $n > N$. Then, for all $n > N$,

$$|\sqrt{t_n} - \sqrt{\ell}| = \frac{1}{\sqrt{\ell}}\,|t_n - \ell| < \frac{1}{\sqrt{\ell}} \cdot \varepsilon\sqrt{\ell} = \varepsilon.$$

This proves that $\lim\limits_{n\to\infty} \sqrt{t_n} = \sqrt{\ell}$.

3.28 According to the arithmetic rules for limits of sequences, $t_n^2 \to \ell^2$ as $n \to \infty$.

Then, in view of Exercise 27, $\sqrt{t_n^2} \to \sqrt{\ell^2}$ as $n \to \infty$. This means that $|t_n| \to |\ell|$ as $n \to \infty$.

Alternative proof by using the definition.

Let $\varepsilon > 0$. Because $\lim_{n\to\infty} t_n = \ell$, there exists an $N \in \mathbb{R}$ such that $|t_n - \ell| < \varepsilon$ for all $n > N$.

Then, according to the Reverse Triangle Inequality, for all $n > N$,

$$\big||t_n| - |\ell|\big| \leq |t_n - \ell| < \varepsilon.$$

This proves that $\lim_{n\to\infty} |t_n| = |\ell|$.

3.29 (c) We will prove that $\lim_{n\to\infty} \frac{n+1}{n\sqrt{n}} = 0$.

Let $\varepsilon > 0$. Note that, for all $n \in \mathbb{N}$,

$$\left|\frac{n+1}{n\sqrt{n}}\right| = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}} \leq \frac{1}{\sqrt{n}} + \frac{1}{\sqrt{n}} = \frac{2}{\sqrt{n}}.$$

Since $\frac{2}{\sqrt{n}} < \varepsilon \iff n > \frac{4}{\varepsilon^2}$, we choose $N = \frac{4}{\varepsilon^2}$. Then, for all $n > N$,

$$\left|\frac{n+1}{n\sqrt{n}}\right| = \frac{1}{\sqrt{n}} + \frac{1}{n\sqrt{n}} \leq \frac{2}{\sqrt{n}} < \varepsilon.$$

This proves that $\lim_{n\to\infty} \frac{n+1}{n\sqrt{n}} = 0$.

3.37 We will prove that the sequence $\big(\ln x_n\big)_{n=1}^{\infty}$ is unbounded. So let $u > 0$.

As

$$\ln x_n < -u \iff x_n < \mathrm{e}^{-u},$$

we choose $\varepsilon = \mathrm{e}^{-u}$. As $\lim_{n\to\infty} x_n = 0$, an N exists such that

$$|x_n| < \varepsilon = \mathrm{e}^{-u},$$

whenever $n > N$. Then for $n > N$,

$$|x_n| < \mathrm{e}^{-u} \iff -\mathrm{e}^{-u} < x_n < \mathrm{e}^{-u}.$$

Hence, for $n > N$, $\ln x_n < -u$.

As the sequence $\big(\ln x_n\big)_{n=1}^{\infty}$ is unbounded, ist is divergent.

3.38 Observe that for all n,

$$1 - \frac{\cos x_n}{n} \geq 1 - \frac{1}{n}.$$

So if $\frac{2n^2}{n^2+\pi}\left[1 - \frac{1}{n}\right] \geq 1.99$, then the value of Babs' investment is at least 1.99α Euro.

Now

$$\begin{aligned}
\frac{2n^2}{n^2+\pi}\left[1-\frac{1}{n}\right] > 1.99 &\iff \frac{2n^2}{n^2+\pi}\frac{n-1}{n} \geq 1.99 \\
&\iff 2n(n-1) \geq 1.99(n^2+\pi) \\
&\iff 2n^2 - 2n - 1.99n^2 - 1.99\pi \geq 0 \\
&\iff 0.01n^2 - 2n - 1.99\pi \geq 0 \\
&\iff n^2 - 200n - 199\pi \geq 0 \\
&\Longleftarrow n \geq \frac{200 + \sqrt{200^2 - 4 \times -199\pi}}{2} = 100 + \sqrt{10\,000 + 199\pi} \approx 203.08.
\end{aligned}$$

So after 204 months her investment almost doubled.