4.1 For any $n \in \mathbb{N}$, $r^{n+1} = \underbrace{r}_{>1} \cdot \underbrace{r^n}_{>0} > r^n$. So the sequence $\big(r^n\big)_{n=1}^{\infty}$ is increasing if $r > 1$.

4.2 (a) Since all terms of the sequence $\big(b_n\big)_{n=1}^{\infty}$ are positive, the sequence is bounded below by 0. According to Example 2, this sequence is decreasing. Hence, by the Monotone Sequence Property, the sequence converges.

(b) For $n \in \mathbb{N}, n \geq 6$ we introduce the statement $\mathcal{P}(n)$: $\dfrac{2^n}{n!} < \dfrac{1}{n}$.

(1) First we show that the statement $\mathcal{P}(6)$ is true: $\dfrac{2^6}{6!} = \dfrac{4}{45} = \dfrac{8}{90} < \dfrac{15}{90} = \dfrac{1}{6}$.

(2) Let $k \in \mathbb{N}, k \geq 6$.

Assume that $\mathcal{P}(k)$ is true, that is: $\dfrac{2^k}{k!} < \dfrac{1}{k}$.

Then

$$\frac{2^{k+1}}{(k+1)!} = \frac{2}{k+1} \cdot \frac{2^k}{k!} < \frac{2}{k+1} \cdot \frac{1}{k} = \underbrace{\frac{2}{k}}_{\leq \frac{1}{3}} \frac{1}{k+1} < \frac{1}{k+1}.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}, n \geq 6$.

(c) Since for all natural numbers $n \geq 6$

$$0 < b_n < \frac{1}{n}$$

and $\lim\limits_{n\to\infty} \dfrac{1}{n} = 0$, the Sandwich Lemma implies that the sequence $\big(b_n\big)_{n=6}^{\infty}$ converges to 0. So $\lim\limits_{n\to\infty} b_n = 0$.

4.3 (a) For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $t_n > \sqrt{2}$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $t_1 = 4 > \sqrt{2}$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $t_k > \sqrt{2}$.

Then

$$t_{k+1} - \sqrt{2} = \tfrac{1}{2}\big(t_k + \frac{2}{t_k}\big) - \sqrt{2} = \frac{t_k^2 + 2 - 2t_k\sqrt{2}}{2t_k} = \frac{\big(t_k - \sqrt{2}\big)^2}{2t_k} > 0.$$

Here we used the fact that $t_k > \sqrt{2} > 0$.

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

(b) Let $n \in \mathbb{N}$. Then

$$t_{n+1} - t_n = \tfrac{1}{2}\big(t_n + \frac{2}{t_n}\big) - t_n = \frac{1}{t_n} - \tfrac{1}{2}t_n = \frac{2 - t_n^2}{2t_n} < 0,$$

where the inequality is a consequence of part (a).

Hence, the sequence is decreasing.

(c) Since the sequence $\big(t_n\big)_{n=1}^{\infty}$ is decreasing and bounded below, the Monotone Sequence Property implies that the sequence converges, say to ℓ. Then $\lim\limits_{n\to\infty} t_{n+1} = \ell$ and $\ell \geq \sqrt{2}$ (because all the terms of the sequence are larger than $\sqrt{2}$). Hence, according to the Arithmetic Rules for limits of sequences,

$$\ell = \lim_{n\to\infty} t_{n+1} = \tfrac{1}{2}\big(\lim_{n\to\infty} t_n + \frac{2}{\lim\limits_{n\to\infty} t_n}\big) = \tfrac{1}{2}\big(\ell + \frac{2}{\ell}\big).$$

So $\ell^2 = 2$, which implies that $\ell = \sqrt{2}$ ($\ell = -\sqrt{2}$ is impossible, because we know that $\ell \geq \sqrt{2}$!).

4.5 (a) For $n \in \mathbb{N}$

$$b_{n+1} = \left(1+\frac{1}{n}\right)^{n+1} = \left(1+\frac{1}{n}\right)^{n}\left(1+\frac{1}{n}\right) = \left(1+\frac{1}{n}\right)a_n.$$

(b) In view of part (a), the sequence $\left(b_n\right)_{n=2}^{\infty}$ is the product of the convergent sequences $\left(a_n\right)_{n=2}^{\infty}$ and $\left(1+\frac{1}{n}\right)_{n=2}^{\infty}$. According to the Arithmetic Rules for limits of sequences, the limit of the sequence $\left(b_n\right)_{n=2}^{\infty}$ is equal to e.

4.7 (a) The sequence $t_5, t_7, t_9, \ldots$ can be written in the form $\left(t_{2k+3}\right)_{k=1}^{\infty}$.

(b) The sequence $t_2, t_4, t_8, t_{16}, \ldots$ can be written in the form $\left(t_{2^k}\right)_{k=1}^{\infty}$.

4.8 Note that we are dealing with a subsequence of the sequence $\left(t_n\right)_{n=1}^{\infty}$, where

$$t_n = \frac{1}{\sqrt{n}} \qquad (n \in \mathbb{N}).$$

Since for $k \in \mathbb{N}$, $t_{k^2} = \dfrac{1}{\sqrt{k^2}} = \dfrac{1}{k}$, the sequence $1, \frac{1}{2}, \frac{1}{3}, \ldots$ is in fact the subsequence $\left(t_{k^2}\right)_{k=1}^{\infty}$.

4.9 (a) The subsequence consisting of the terms with the numbers $1, 3, 6, 10, 15, \ldots$ is a constant sequence: all terms are equal to 1 (take $n_k = \frac{1}{2}k(k+1)$ in the definition of subsequence). Hence, this subsequence converges to 1.

(b) The subsequence consisting of the terms with the numbers $2, 4, 7, 11, 11, \ldots$ is the sequence $\frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ (take $n_k = 1+\frac{1}{2}k(k+1)$ in the definition of subsequence). This subsequence converges to 0. So the given sequence has two convergent subsequences with different limits. Hence the given sequence is divergent.

4.11 (a) Note that

$$t_n = n + (-1)^n n = \begin{cases} 2n & \text{if } n \text{ is even} \\ 0 & \text{if } n \text{ is odd.} \end{cases}$$

Hence, the subsequence $\left(t_{2n-1}\right)_{n=1}^{\infty}$ converges to 0.

(b) According to part (a), the subsequence $\left(t_{2n}\right)_{n=1}^{\infty}$ is in fact the increasing sequence

$$4, 8, 12, 16, \ldots$$

Obviously, the subsequence $\left(t_{4n}\right)_{n=1}^{\infty}$ is a subsequence of the foregoing one. Hence, this subsequence is increasing too.

4.14 Let $\varepsilon > 0$. Then an N_1 exists such that

$$|a_{2n} - \ell| < \varepsilon,$$

whenever $2n > N_1$. Likewise, an N_2 exists such that

$$|a_{2n-1} - \ell| < \varepsilon,$$

whenever $2n - 1 > N_2$.

We choose $N = \max\{N_1, N_2\}$. Then, for all $k > N$,

$$|a_k - \ell| < \varepsilon.$$

This can be explained as follows:

– if k is even, then k can be written as $k = 2n$ for some $n \in \mathbb{N}$; because $2n > N \geq N_1$, it follows that

$$|a_k - l| = |a_{2n} - \ell| < \varepsilon;$$

– if k is odd, then k can be written as $k = 2n - 1$ for some $n \in \mathbb{N}$; because $2n - 1 > N \geq N_2$, it follows that

$$|a_k - l| = |a_{2n-1} - \ell| < \varepsilon.$$

Hence, $\lim_{n \to \infty} a_n = \ell$.