3.34 We give a proof by contradiction. So assume that $\ell \neq \ell'$.

Let $\varepsilon = |\ell - \ell'|$.

As $\lim\limits_{n\to\infty} t_n = \ell$, there exist an N such that

$$|t_n - \ell| < \tfrac{1}{2}\varepsilon,$$

whenever $n > N$. Similarly, as $\lim\limits_{n\to\infty} t_n = \ell'$, there exist an N' such that

$$|t_n - \ell'| < \tfrac{1}{2}\varepsilon,$$

whenever $n > N'$.

Hence, for any $n > \max\{N, N'\}$, the Triangle Inequality implies that

$$\varepsilon = |\ell - \ell'| = |\ell - t_n + t_n - \ell'| \leq |\ell - t_n| + |t_n - \ell'| < \tfrac{1}{2}\varepsilon + \tfrac{1}{2}\varepsilon = \varepsilon,$$

which is impossible.

3.37 We will prove that the sequence $\big(\ln x_n\big)_{n=1}^{\infty}$ is unbounded. So let $u > 0$.

As

$$\ln x_n < -u \iff x_n < \mathrm{e}^{-u},$$

we choose $\varepsilon = \mathrm{e}^{-u}$. As $\lim\limits_{n\to\infty} x_n = 0$, an N exists such that

$$|x_n| < \varepsilon = \mathrm{e}^{-u},$$

whenever $n > N$. Then for $n > N$,

$$|x_n| < \mathrm{e}^{-u} \iff -\mathrm{e}^{-u} < x_n < \mathrm{e}^{-u}.$$

Hence, for $n > N$, $\ln x_n < -u$.

As the sequence $\big(\ln x_n\big)_{n=1}^{\infty}$ is unbounded, it is divergent.

4.4 Note that for $n > 1$

$$\frac{a_n}{a_{n-1}} = \frac{\left(1+\dfrac{1}{n}\right)^n}{\left(1+\dfrac{1}{n-1}\right)^{n-1}} = \left(\frac{1+\dfrac{1}{n}}{1+\dfrac{1}{n-1}}\right)^n \left(1+\frac{1}{n-1}\right) = \left(\frac{(n+1)(n-1)}{n^2}\right)^n \cdot \frac{n}{n-1}$$

$$= \left(1-\frac{1}{n^2}\right)^n \cdot \frac{n}{n-1} \underset{\substack{\text{Bernoulli's}\\ \text{Inequality}}}{\geq} \left(1 - n\cdot\frac{1}{n^2}\right)\cdot\frac{n}{n-1} = \frac{n-1}{n}\cdot\frac{n}{n-1} = 1.$$

As the terms of the sequence are positive, this implies that the sequence is increasing.

4.6 (a) For any $n \in \mathbb{N}$,

$$1 + \frac{2}{n} = \frac{n+2}{n} = \frac{n+1}{n}\,\frac{n+2}{n+1} = \left(1+\frac{1}{n}\right)\left(1+\frac{1}{n+1}\right).$$

(b) According to part (a), for any n,

$$\left(1+\frac{2}{n}\right)^n = \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1}\right)^n = \left(1+\frac{1}{n}\right)^n \frac{\left(1+\dfrac{1}{n+1}\right)^{n+1}}{1+\dfrac{1}{n+1}}$$

$$= \left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{n+1}\right)^{n+1}\frac{n+1}{n+2}.$$

As the limits of the first two factors of this expression are both equal to e and the limit of the third factor is 1, the Product Rule for limits of sequences implies that the limit of the expression is equal to e^2.

4.10 Assume that $\lim\limits_{n\to\infty}(-1)^n b_n = \ell$.

Then, according to Theorem 2,

$$\ell = \lim_{k\to\infty}(-1)^{2k}b_{2k} = \lim_{k\to\infty}\underbrace{b_{2k}}_{\geq 0} \underset{\text{Theorem 3.1}}{\geq} 0.$$

Similarly,

$$\ell = \lim_{k\to\infty}(-1)^{2k-1}b_{2k-1} = -\lim_{k\to\infty}b_{2k-1} \leq 0.$$

So $\ell = 0$. Now, according to Exercise 3.28, the sequence $\left(b_n\right)_{n=1}^{\infty} = \left(\left|(-1)^n b_n\right|\right)_{n=1}^{\infty}$ converges to $|\ell| = 0$.

4.12 (a) For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $2 \leq t_n \leq 3$.

(1) First we show that the statement $\mathcal{P}(1)$ is true: $2 \leq 2 \leq 3$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $2 \leq t_k \leq 3$.

Then

$$7 \leq 3+2t_k \leq 9 \Longrightarrow 2 < \sqrt{7} \leq \sqrt{3+2t_k} \leq \sqrt{9} = 3 \Longrightarrow 2 < t_{k+1} \leq 3.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

(b) Let $n \in \mathbb{N}$. Then the fact that $2 \leq t_n \leq 3$ implies that

$$t_{n+1} - t_n = \sqrt{3+2t_n} - t_n = \frac{[\sqrt{3+2t_n} - t_n][\sqrt{3+2t_n} + t_n]}{\sqrt{3+2t_n} + t_n} = \frac{3+2t_n - t_n^2}{\sqrt{3+2t_n}+t_n}$$

$$= \frac{-(t_n-3)(t_n+1)}{\sqrt{3+2t_n}+t_n} \geq 0.$$

Hence, the sequence $\left(t_n\right)_{n=1}^{\infty}$ is increasing.

Alternative:

As t_n and t_{n+1} are both positive,

$$t_{n+1} \geq t_n \Longleftrightarrow t_{n+1}^2 \geq t_n^2 \Longleftrightarrow 3+2t_n \geq t_n^2 \Longleftrightarrow t_n^2 - 2t_n - 3 \leq 0 \Longleftrightarrow (t_n-3)(t_n+1) \leq 0$$

$$\Longleftrightarrow -1 \leq t_n \leq 3.$$

So, according to part (a), the sequence $\left(t_n\right)_{n=1}^{\infty}$ is increasing.

(c) Because the sequence is increasing and bounded above (by 3), the Monotone Sequence Property implies the convergence of the sequence $\left(t_n\right)_{n=1}^{\infty}$. Let ℓ be the limit of the sequence. Then $\lim\limits_{n\to\infty} t_{n+1} = \ell$ and, in view of part (a), $2 \leq \ell \leq 3$.

So, according to the Arithmetic Rules for limits of sequences,

$$\ell = \lim_{n\to\infty} t_{n+1} = \lim_{n\to\infty} \sqrt{3+2t_n} = \sqrt{3+2\ell}.$$

Hence,

$$\ell^2 = 3 + 2\ell \iff (\ell-3)(\ell+1) = 0 \iff \ell = 3 \text{ or } \ell = -1.$$

So $\ell = 3$.

4.13 Note that for any $n \in \mathbb{N}$,

$$\left(1+\frac{1}{2n}\right)^n = \left[\left(1+\frac{1}{2n}\right)^{2n}\right]^{\frac{1}{2}} = \sqrt{\left(1+\frac{1}{2n}\right)^{2n}}.$$

As the limit of the expression after the square root is equal to e (this expression corresponds to the subsequence of the even-numbered terms of the sequence defining the number e), Exercise 3.27 implies that the limit of the expression is equal to $\mathrm{e}^{\frac{1}{2}}$.