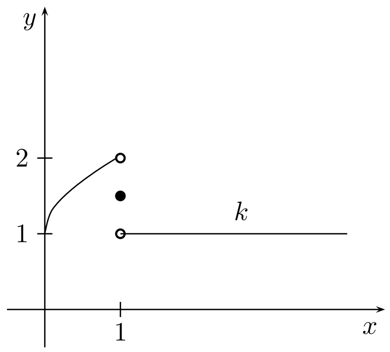
5.1 If we consider the graph of the function k, then we see that the graph has a 'hole' at $x = 1$ and that it makes a 'jump' at 1. So the function is not continuous at $x = 1$.



5.2 Let $\left(x_n\right)_{n=1}^{\infty}$ be a sequence converging to 3.

We have to prove that $\lim_{n\to\infty} f(x_n) = f(3) = 21$.

According to the Arithmetic Rules for limits of sequences,

$$\lim_{n\to\infty} f(x_n) = \lim_{n\to\infty} \left[x_n^2 + 2x_n + 6\right] = \lim_{n\to\infty} \left[x_n^2\right] + \lim_{n\to\infty} \left[2x_n\right] + \lim_{n\to\infty} 6$$
$$= \left[\lim_{n\to\infty} x_n\right]^2 + 2 \lim_{n\to\infty} x_n + 6 = 9 + 6 + 6 = 21.$$

An ε, N-proof goes as follows.

Let $\varepsilon > 0$. We must find an N such that

$$\left|f(x_n) - 21\right| = \left|x_n^2 + 2x_n + 6 - 21\right| < \varepsilon,$$

whenever $n > N$. Note, however, that for all n,

$$\left|x_n^2 + 2x_n + 6 - 21\right| = \left|x_n^2 + 2x_n - 15\right| = \left|(x_n + 5)(x_n - 3)\right| = |x_n + 5||x_n - 3|.$$

Since the sequence $\left(x_n\right)_{n=1}^{\infty}$ converges to 3, we can find, for $\varepsilon = 1$ a number $N' \in \mathbb{R}$ such that $|x_n - 3| < 1$, whenever $n > N'$. Then for all $n > N'$,

$$|x_n + 5| = |x_n - 3 + 8| \leq |x_n - 3| + 8 < 9,$$

so that

$$\left|x_n^2 + 2x_n + 6 - 21\right| = |x_n + 5||x_n - 3| < 9|x_n - 3|.$$

Since $x_n \to 3$ as $n \to \infty$, an N'' exists such that

$$|x_n - 3| < \tfrac{1}{9}\varepsilon,$$

whenever $n > N''$. Hence, for $n > N = \max\{N', N''\}$,

$$\left|x_n^2 + 2x_n + 6 - 21\right| < 9|x_n - 3| < \varepsilon.$$

In other words: $\lim_{n\to\infty} \left(x_n^2 + 2x_n + 6\right) = 21$.

As the sequence $\left(x_n\right)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to 3} f(x) = 21$.

5.4 Let $\big(t_n\big)_{n=1}^{\infty}$ be a sequence of nonzero terms converging to 0.

We have to prove that $\lim_{n\to\infty} q(t_n) = q(0) = 6$. Note that for all n,

$$q(t_n) = \frac{(2+3t_n)^2-4}{2t_n} = \frac{4+12t_n+9t_n^2-4}{2t_n} = 6 + 4\tfrac{1}{2}t_n.$$

Here we used the fact that $t_n \neq 0$, for all n.

Hence, the Arithmetic Rules for limits of sequences imply that

$$\lim_{n\to\infty} q(t_n) = \lim_{n\to\infty}\big[6 + 4\tfrac{1}{2}t_n\big] = 6 + 4\tfrac{1}{2}\lim_{n\to\infty} t_n = 6.$$

An ε, N-proof goes as follows.

Let $\varepsilon > 0$. We must find an N such that

$$\big|f(x_n) - 6\big| = \Big|\frac{(2+3x_n)^2-4}{2x_n} - 6\Big| < \varepsilon,$$

whenever $n > N$. Note, however, that for all n,

$$\Big|\frac{(2+3x_n)^2-4}{2x_n} - 6\Big| = \Big|\frac{4+12x_n+9x_n^2-4}{2x_n} - \frac{12x_n}{2x_n}\Big| = \tfrac{9}{2}|x_n|.$$

Since $x_n \to 0$ as $n \to \infty$, an N exists such that

$$|x_n| < \tfrac{2}{9}\varepsilon,$$

whenever $n > N$. Hence, for $n > N$,

$$\Big|\frac{(2+3x_n)^2-4}{2x_n} - 6\Big| = \tfrac{9}{2}|x_n| < \varepsilon.$$

In other words: $\lim_{n\to\infty} \dfrac{(2+3x_n)^2-4}{2x_n} = 6$.

As the sequence $\big(x_n\big)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to 0} f(x) = 6$.

5.9 Let $\big(x_n\big)_{n=1}^{\infty}$ be a sequence with positive terms converging to 0.

Let $\varepsilon > 0$. Because $\lim_{n\to\infty} x_n = 0$, there exists an $N \in \mathbb{R}$ such that

$$\underbrace{|x_n - 0|}_{=x_n} < \varepsilon^3,$$

whenever $n > N$. Then, for all $n > N$,

$$|\sqrt[3]{x_n} - 0| = \sqrt[3]{x_n} < \sqrt[3]{\varepsilon^3} = \varepsilon.$$

This proves that $\lim_{n\to\infty} \sqrt[3]{x_n} = 0 = f(0)$.

Hence, the function f is continuous at 0.

Now let $c \in (0,1)$. We will show that f is continuous at $x = c$.

Observe that

$$f(c) = \sqrt[3]{c} = \sqrt[3]{\frac{1}{\frac{1}{c}}} = \frac{1}{\sqrt[3]{\frac{1}{c}}}.$$

If $\big(x_n\big)_{n=1}^{\infty}$ is a sequence with positive terms converging to c, then

$$\frac{1}{x_n} \to \frac{1}{c} \quad \text{as} \quad n \to \infty.$$

As $\dfrac{1}{c} > 1$, Exercise 7 implies that

$$\sqrt[3]{\frac{1}{x_n}} \to \sqrt[3]{\frac{1}{c}} \quad \text{as} \quad n \to \infty,$$

so that

$$\sqrt[3]{x_n} = \frac{1}{\sqrt[3]{\dfrac{1}{x_n}}} \to \frac{1}{\sqrt[3]{\dfrac{1}{c}}} = \sqrt[3]{c} \quad \text{as} \quad n \to \infty.$$

5.10 (b) According to Exercise 4, the function q is continuous at $t = 0$.

Let $c \neq 0$. We show that q is continuous at $t = c$.

In view of Example 6, the (polynomial) function $t \mapsto 12t + 9t^2$ is continuous at c. Similarly, the function $t \mapsto 2t$ is continuous at c. So the Quotient Rule for continuous functions implies that the function q is continuous at $t = c$. Since $c \neq 0$ was arbitrarily chosen, the function q is continuous.

5.11 Let $c > 0$. By Example 5, the function g on $(0, \infty)$ defined by $g(x) = x^n$ is continuous at c.

Since $g(x) \neq 0$, for all $x > 0$, the Quotient Rule for continuous functions implies that the function $f = \dfrac{1}{g}$ is continuous at c.

Since the number $c > 0$ was chosen arbitrarily, the function f is continuous on the interval $(0, \infty)$.

In a similar way one can prove that the function f is continuous on the interval $(-\infty, 0)$.

5.12 Let $c \in J$ and let $\big(x_n\big)_{n=1}^{\infty}$ be a sequence in J converging to c. Since g is continuous at c, $\lim_{n\to\infty} g(x_n) = g(c)$.

Then, according to Exercise 3.28,

$$\lim_{n\to\infty} |g|(x_n) = \lim_{n\to\infty} |g(x_n)| = |g(c)| = |g|(c).$$

As the sequence $\big(x_n\big)_{n=1}^{\infty}$ was chosen arbitrarily, this proves that $\lim_{x\to c} |g|(x) = |g|(c)$. In other words: the function $|g|$ is continuous at c.

As c was chosen arbitrarily, this implies that the function g is continuous.

Alternative: first prove that the function $f\colon x \to |x|$ is continuous and then apply Theorem 2.

5.17 (b) As $-1 \leq \sin t \leq 1$ for all t, we get $|\sin t| \leq 1$. This implies that for all $x \neq 0$,

$$\big|x \sin \frac{1}{x}\big| = |x| \, \big|\sin \frac{1}{x}\big| \leq |x|.$$

As a consequence, for all $x \neq 0$,

$$-|x| \leq x \sin \frac{1}{x} \leq |x|,$$

so that for all x, $-|x| \leq f(x) \leq |x|$. As the functions h and g defined by $g(x) = -|x|$ and $h(x) = |x|$, are continuous at $x = 0$ and $g(0) = h(0)$, the Sandwich Lemma implies that the function f is continuous at $x = 0$.

5.18 We introduce the function f on the interval $[1,2]$ by $f(x) = x^3 + 2$ and the function h on the interval $[0,1]$ by $h(x) = x(x+2) = x^2 + 2x$. According to Example 6 these functions are continuous at $x = 1$. As $f(1) = 3 = h(1)$, the Glue Lemma implies that the function g is continuous at $x = 1$.

5.21 As $g(2) = 9$,

$$|g(x) - g(2)| < \tfrac{1}{1000} \Longleftrightarrow |x^2 + 2x + 1 - 9| < \tfrac{1}{1000} \Longleftrightarrow |x^2 + 2x - 8| < \tfrac{1}{1000}$$

$$\Longleftrightarrow |(x+4)(x-2)| < \tfrac{1}{1000} \Longleftrightarrow |x+4|\,|x-2| < \tfrac{1}{1000}.$$

If we choose, in advance, $\delta < 1$, then for $x \in (2-\delta, 2+\delta)$,

$$2 - \delta < x < 2 + \delta \Longrightarrow 6 - \delta < x + 4 < 6 + \delta \Longrightarrow 5 < x + 4 < 7 \Longrightarrow |x+4| < 7.$$

Hence, the inequality $|g(x) - g(2)| < \frac{1}{1000}$ is satisfied if

$$|x - 2| < \tfrac{1}{7000}.$$

So we take $\delta = \frac{1}{7000}$ (which is smaller than 1).
For an arbitrary $\varepsilon > 0$ we take $\delta = \min\{1, \frac{1}{7}\varepsilon\}$. Then for any x in the interval $2-\delta, 2+\delta)$, we know that $|x+4| < 7$ (because $\delta \le 1$) and that $|x-2| < \frac{1}{7}\varepsilon$ (because $\delta \le \frac{1}{7}\varepsilon$).
Hence,

$$|g(x) - g(2)| = |x+4|\,|x-2| < 7 \cdot \tfrac{1}{7}\varepsilon = \varepsilon.$$

That is: the function g is continuous at $x = 2$.

5.22 Observe that

$$|k(x)| < \tfrac{1}{1000} \Longleftrightarrow |x(x-1)(x+2)| < \tfrac{1}{1000} \Longleftrightarrow |x-1| \cdot |x(x+2)| < \tfrac{1}{1000}.$$

If we choose, in advance, $\delta \le 1$, then for $x \in (1-\delta, 1+\delta)$, $0 < x < 2$, so that

$$|x(x+2)| = |x^2 + 2x| \le |x^2| + |2x| = x^2 + 2|x| \le 4 + 4 = 8.$$

So the above inequality is satisfied when $|x-1| < \frac{1}{8000}$. Hence, we may choose $\delta = \frac{1}{8000}$.