5.3 We prove that the function h defined by

$$h(x) = \begin{cases} \dfrac{2\sqrt{x}-4}{x-4} & \text{if } x \neq 4 \\ \frac{1}{2} & \text{if } x = 4. \end{cases}$$

is continuous at $x = 4$.

Let $(x_n)_{n=1}^{\infty}$ be a sequence converging to 4. We assume that $x_n \neq 4$ for all n (Can you explain why this is allowed?). We show that

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty} \frac{2\sqrt{x_n}-4}{x_n-4} = \tfrac{1}{2}.$$

Since $\lim_{n\to\infty}(x_n-4) = 0$, we cannot use the Quotient Rule for limits of sequences directly, but if we apply the simplification (here we use that $x_n \neq 4$)

$$\frac{2\sqrt{x_n}-4}{x_n-4} = \frac{2(\sqrt{x_n}-2)}{(\sqrt{x_n}-2)(\sqrt{x_n}+2)} = \frac{2}{\sqrt{x_n}+2},$$

we can use the Quotient Rule (and Exercise 3.27) to obtain

$$\lim_{n\to\infty} \frac{2\sqrt{x_n}-4}{x_n-4} = \lim_{n\to\infty} \frac{2}{\sqrt{x_n}+2} = \tfrac{1}{2}.$$

As the sequence $(x_n)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to 4} \dfrac{2\sqrt{x}-4}{x-4} = \frac{1}{2} = h(4)$. That is: the function h is continuous at $x = 4$.

5.5 Let $c > 0$. We will prove that the function f is continuous at c.

Let $(x_n)_{n=1}^{\infty}$ be a sequence converging to c. We have to prove that $\lim_{n\to\infty} f(x_n) = f(c) = \dfrac{1}{c}$.

Since $x_n \to c \neq 0$ as $n \to \infty$, the Arithmetic Rules for limits of sequences imply that

$$\lim_{n\to\infty} \frac{1}{x_n} = \frac{1}{\lim_{n\to\infty} x_n} = \frac{1}{c}.$$

As the sequence $(x_n)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to c} f(x) = f(c)$.

5.6 Take an arbitrary $c \in \mathbb{R}$ and let $(y_n)_{n=1}^{\infty}$ be a sequence converging to c.

We have to show that $\lim_{n\to\infty} t(y_n) = t(c)$.

Since $y_n \to c$ as $n \to \infty$, the Arithmetic Rules for limits of sequences imply that

$$\lim_{n\to\infty} t(y_n) = \lim_{n\to\infty} \frac{5y_n^7 - y_n^2 + 15}{y_n^2+7} = \frac{5\lim_{n\to\infty} y_n^7 - \lim_{n\to\infty} y_n^2 + 15}{\lim_{n\to\infty} y_n^2 + 7} = \frac{5c^7 - c^2 + 15}{c^2+7} = t(c).$$

As the sequence $(y_n)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that t is continuous at c.

5.7 Let $(x_n)_{n=1}^{\infty}$ be a sequence in $I \setminus \{c\}$ converging to c.

Then $\lim_{n\to\infty} f(x_n) = f(c)$.

Since $f(x_n) \geq 0$ for all n, Theorem 3.1 implies that $f(c) \geq 0$.

5.10 (a) According to Exercise 3, the function h is continuous at $x = 4$.

Now let $c \geq 0$ and $c \neq 4$. We will prove that the function h is continuous at c.

According to Example 2, the function $x \mapsto \sqrt{x}$ is continuous at c. So the Product Rule for continuous functions implies that the function $x \mapsto 2\sqrt{x}$ is continuous at c (here we use the fact that the constant function $x \mapsto 2$ is continuous). By the Sum Rule of continuous functions, the function $x \mapsto 2\sqrt{x} - 4$ is continuous at c.

Example 6 implies that the function $x \mapsto x - 4$ is continuous at c.

Finally, the Quotient Rule for continuous functions implies that the function h is continuous at c.

Since c was chosen arbitrarily, the function h is continuous.

5.13 First note that $g(c) = f(c) = h(c)$.

Let $\left(x_n\right)_{n=1}^{\infty}$ be a sequence in I converging to c.

Then the continuity of g and h at c implies that

$$\lim_{n\to\infty} h(x_n) = h(c) = f(c) = g(c) = \lim_{n\to\infty} g(x_n).$$

Furthermore for all $n \in \mathbb{N}$

$$g(x_n) \leq f(x_n) \leq h(x_n).$$

By the Sandwich Lemma (for sequences) this implies that $\lim_{n\to\infty} f(x_n) = f(c)$. Because the sequence $\left(x_n\right)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to c} f(x) = f(c)$, that is: f is continuous at c.

5.19 We introduce the function g on the interval $[1, 2]$ by $g(t) = t$ and the function h on the interval $[0, 1]$ by $h(t) = 2t^2 - 1$. According to Example 6 these functions are continuous at $t = 1$. As $g(1) = 1 = h(1)$, the Glue Lemma implies that the function p is continuous at $t = 1$.

5.23 Note that $h(2) = 10$ and that

$$|h(z) - h(2)| < \tfrac{1}{1000} \iff |z^3 + z - 10| < \tfrac{1}{1000}.$$

By deviding $z^3 + z - 10$ by $z - 2$ we find that

$$z^3 + z - 10 = (z - 2)(z^2 + 2z + 5).$$

If we choose, in advance, $\delta < 1$, then for $z \in (2 - \delta, 2 + \delta)$,

$$|z^2 + 2z + 5| = z^2 + 2z + 5 < 9 + 6 + 5 = 20,$$

so that

$$|z^3 + z - 10| < 20|z - 2|.$$

Hence, the inequality $|h(z) - h(2)| < \frac{1}{1000}$ is satisfied if

$$|z - 2| < \tfrac{1}{20\,000}.$$

So we take $\delta = \frac{1}{20\,000}$ (which is smaller than 1).
For an arbitrary $\varepsilon > 0$ we take $\delta = \min\{1, \frac{1}{20}\varepsilon\}$. Then for any z in the interval $(2-\delta, 2+\delta)$, we know that $|z^2 + 2z + 5| < 20$ (because $\delta \leq 1$) and that $|z - 2| < \frac{1}{20}\varepsilon$ (because $\delta \leq \frac{1}{20}\varepsilon$).
Hence,

$$|h(z) - h(2)| = |(z-2)(z^2 + 2z + 5)| < 20|z - 2| < \varepsilon.$$

That is: the function h is continuous at $z = 2$.