5.14 Let $\left(x_n\right)_{n=1}^{\infty}$ be a sequence with positive terms which converges to 0. Then, according to the Arithmetic Rules for limits of sequences,

$$f(x_n) = \frac{x_n^2 + \sqrt{x_n}}{x_n + \sqrt{x_n}} = \frac{x_n\sqrt{x_n} + 1}{\sqrt{x_n} + 1} = \frac{1}{1} = 1 = f(0).$$

As the sequence $\left(x_n\right)_{n=1}^{\infty}$ was chosen arbitrarily, this implies that the function f is continuous at $x = 1$.

5.20 Note that

$$u(x) = \max\{-5x, 3x\} = \begin{cases} 3x & \text{if } x \geq 0 \\ -5x & \text{if } x < 0. \end{cases}$$

The function g on $[0, 1]$ defined by $g(x) = 3x$ is continuous at $x = 0$. The function h on $[-1, 0]$ defined by $h(x) = -5x$ is continuous at $x = 0$. As $g(0) = 0 = h(0)$, the Glue Lemma implies that the function u is continuous at $x = 0$.

5.27 As the function is continuous at c, the Linking Limit Lemma implies that for every $\varepsilon > 0$, a $\delta > 0$ exists such that

$$|f(x) - f(c)| < \varepsilon,$$

for any x in the interval $(c - \delta, c + \delta)$.

Now let $\varepsilon = f(c)$. Then a $\delta > 0$ exists such that for any x in the interval $(c - \delta, c + \delta)$,

$$|f(x) - f(c)| < f(c) \Longrightarrow -f(c) < f(x) - f(c) < f(c) \Longrightarrow 0 < f(x) < 2f(c).$$

6.1 Assume that $|f(x)| \leq m$ for all $x \in [a, b]$. Then for all $x \in [a, b]$, $-m \leq f(x) \leq m$. So the function is bounded above by m and bounded below by $-m$, That is: the function is bounded.

Assume that the function is bounded. Then numbers ℓ and u exist such that $\ell \leq f(x) \leq u$ for all $x \in [a, b]$. Then for all $x \in [a, b]$,

$$-\max\{|\ell|, |u|\} \leq |\ell| \leq \ell \leq f(x) \leq u \leq |u| \leq \max\{|\ell|, |u|\}.$$

So if we choose $m = 1 + \max\{|\ell|, |u|\}$, then $m > 0$ and $|f(x)| \leq m$, for all $x \in [a, b]$.

6.2 Because the function f is continuous on the compact interval $[a, b]$, the Theorem of Weierstrass implies that a $c \in [a, b]$ exists such that $f(x) \geq f(c)$ for all $x \in [a, b]$.

Choose $m = f(c)$. Then $m = f(c) > 0$, because $c \in [a, b]$.

6.3 The inequality $f(a)f(b) < 0$ implies that $f(a)$ and $f(b)$ have an opposite sign (and are non-zero). So 0 is a value between $f(a)$ and $f(b)$.

According to the Intermediate Value Theorem, there exists a $z \in (a, b)$ such that $f(z) = 0$.

6.4 According to Example 6, the function f is continuous.

Note that the restriction of the function f to the interval $[0, 2]$ is continuous, that $f(0) = 16 > 0$ and that $f(2) = -6 < 0$. So by the Intermediate Value Theorem (applied to the restriction of the function f to the interval $[0, 2]$) a $\tau \in (0, 2)$ exists satisfying $f(\tau) = 0$.

The interval $(2, 4)$ can be treated in a similar way.

Note that the restriction of the function f to the interval $[-10, 0]$ is continuous, that $f(0) = 16 > 0$ and that $f(-10) < 0$. So by the Intermediate Value Theorem (applied to the restriction of the function f to the interval $[-10, 0]$) a $\tau \in (-10, 0)$ exists satisfying $f(\tau) = 0$.

6.6 Let f be the function on $(0, 1)$ defined by

$$f(x) = x^2.$$

Then $0 < f(x) < 1$ for all $x \in (0, 1)$. The function f is continuous but there doesn't exist a $c \in (0, 1)$ such that $f(c) = c$:

$$f(x) = x \iff x^2 = x \iff x(x-1) = 0 \iff x = 0 \text{ or } x = 1.$$

6.9 Let $a > 0$. Then

$$f(a) = a^3 + a > a$$

and

$$f(0) = 0 < a.$$

So the Intermediate Value Theorem implies that the equation $f(x) = a$ has a solution, that is: $a \in R_f$.

As $f(0) = 0$, $0 \in R_f$.

Let $a < 0$. Then

$$f(a) = a^3 + a < a$$

and

$$f(0) = 0 > a.$$

So the Intermediate Value Theorem implies that the equation $f(x) = a$ has a solution, that is: $a \in R_f$.

In all: $R_f = \mathbb{R}$.

1.33 Assume that $f(x) = f(x')$ for $x, x' \geq 0$. Then

$$f(x) = f(x') \iff x^2 + 1 = (x')^2 + 1 \iff x^2 = (x')^2 \iff x = x'.$$

So the function is invertible.

Since $f(x) = x^2 + 1 \geq 1$ for all $x \geq 0$, $R_f \subset [1, \infty)$. In order to find f^{-1}, we consider, for $y \geq 1$ and $x \geq 0$, the equation

$$f(x) = y \iff y = x^2 + 1 \iff x^2 = y - 1.$$

Obviously, this equation has a solution if and only if $y - 1 \geq 0 \iff y \geq 1$. Further, for $y \geq 1$,

$$f(x) = y \iff x^2 - 1 \iff x = \sqrt{y-1}.$$

Hence, $D_{f^{-1}} = R_f = [1, \infty)$ and f^{-1} is the function on $[1, \infty)$, defined by

$$f^{-1}(y) = \sqrt{1-y}.$$