1.34 (a) Assume that $f(x) = f(x')$ for $x, x' \geq 0$. Then

$$f(x) = f(x') \iff \frac{x}{\sqrt{1-x^2}} = \frac{x'}{\sqrt{1-(x')^2}} \iff x\sqrt{1-(x')^2} = x'\sqrt{1-x^2}$$
$$\iff x^2\left[1-(x')^2\right] = (x')^2\left[1-x^2\right]$$
$$\iff x^2 - x^2(x')^2 = (x')^2 - x^2(x')^2$$
$$\iff x^2 = (x')^2 \iff x = x'.$$

So the function is invertible.

(b) Since $f(x) \geq 0$ for all $x \in [0,1)$, we consider the equation $f(x) = y$ for $y \geq 0$ and $0 \leq x < 1$:

$$f(x) = y \iff y = \frac{x}{\sqrt{1-x^2}} \iff y^2(1-x^2) = x^2 \iff x^2(1+y^2) = y^2 \iff x^2 = \frac{y^2}{1+y^2}.$$

Obviously, this equation has a solution for any $y \geq 0$. Hence, $D_{f^{-1}} = R_f = [0,\infty)$ and for $y \geq 0$,

$$f(x) = y \iff x^2 = \frac{y^2}{1+y^2} \iff x = \sqrt{\frac{y^2}{1+y^2}}.$$

So and f^{-1} is the function on $[0,\infty)$, defined by

$$f^{-1}(y) = \sqrt{\frac{y^2}{1+y^2}}.$$

5.16 (a) Note that $PS = \sin x$ and that

$$\tan x = \frac{QR}{OQ} = QR.$$

The area of the triangle OPQ equals $\frac{1}{2}PS \cdot OQ = \frac{1}{2}\sin x$. Furthermore, the area of the circular sector OPQ is equal to the area of the circle, namely π, multiplied by $\frac{x}{2\pi}$. Thus the area equals $\frac{1}{2}x$.
The area of the triangle ORQ equals $\frac{1}{2}QR \cdot OQ = \frac{1}{2}\tan x$.
Since the area of the triangle OPQ is smaller than the area of the the circular sector OPQ which is smaller than the area of the triangle ORQ, we have

$$\tfrac{1}{2}\sin x \leq \tfrac{1}{2}x \leq \tfrac{1}{2}\tan x.$$

(b) According to part (a),

$$\sin x \leq x \leq \tan x \Longrightarrow \sin x \leq x \leq \frac{\sin x}{\cos x} \Longrightarrow 1 \leq \frac{x}{\sin x} \leq \frac{1}{\cos x} \Longrightarrow \cos x \leq \frac{\sin x}{x} \leq 1.$$

(c) If $x \in (-\pi/2, 0)$, then $-x \in (0, \pi/2)$. Hence,

$$\cos x = \cos(-x) \leq \frac{\sin(-x)}{-x} \leq 1 \Longrightarrow \cos x \leq \frac{\sin x}{x} \leq 1.$$

(d) Let g and h be the functions on $(-\pi/2, \pi/2)$ defined by $g(x) = \cos x$, and $h(x) = 1$, respectively. As the functions g and h are continuous at 0, and $g \leq f \leq h$ (by parts (b)and (c)), the Sandwich Lemma implies that f is continuous at 0. That is:

$$\lim_{x\to 0}\frac{\sin x}{x} = 1.$$

6.5 Let f be a continuous function on $[a, b]$. Define the function h on the interval $[a, b]$, by $h(x) = f(x) - x$. (We got the idea of introducing the function h by sketching, as in Figure 2, in one and the same coordinate system both the graph of f and the line $y = x$.)

The function h is continuous, $h(a) = f(a) - a \geq 0$ and $h(b) = f(b) - b \leq 0$. According to the Intermediate Value Theorem, applied to the function h and the value 0, an $x^* \in [a, b]$ exists such that $h(x^*) = 0$, in other words: $f(x^*) = x^*$.

6.7 The function f on $[0, 1]$ defined by

$$f(x) = \begin{cases} 0 & \text{if } x > 0 \\ 1 & \text{if } x = 0, \end{cases}$$

is not continuous at 0. Obviously, this function possesses no fixed point.

6.10 We introduce the polynomial function f on the interval $[1, 2]$ defined by

$$f(x) = x^7 + x^5 + x - 4.$$

As $f(1) = -1 < 0$ and $f(2) = 158 > 0$, and the function f is continuous, the Intermediate Value Theorem implies the existence of an $x^* \in (1, 2)$ such that $f(x^*) = 0$.

Let $a, b \in (1, 2)$ with $a < b$. Then Exercise 2.16 implies that

$$f(a) = a^7 + a^5 + a - 4 < b^7 + b^5 + b - 4 = f(b).$$

So if $x^{**} \in (1, 2)$ is another zero of the function f, and $x^* < x^{**}$, then

$$0 = f(x^*) < f(x^{**}) = 0,$$

which is a contradiction. So the function f has a unique zero.

As $f(1) < 0$, $f(1.1) = 1.1^7 + 1.1^5 + 1.1 - 4 = 1.9487171 + 1.61051 + 1.1 - 4 = 0.6592271 > 0$, and because the function f restricted to the interval $[1, 1.1]$ is continuous, the Intermediate Value Theorem implies the existence of an $x^* \in (1, 1.1)$ such that $f(x^*) = 0$.

6.14 The way the exercise has been formulated suggests the use of the Intermediate Value Theorem.

We introduce the function h on $[0, \frac{1}{2}]$, defined by

$$h(x) = f(x) - f(x + \tfrac{1}{2}).$$

Note that

$$h(c) = 0 \iff f(c) - f(c + \tfrac{1}{2}) = 0 \iff f(c) = f(c + \tfrac{1}{2}).$$

We will show that the function h has a zero.

We will assume that $h(0) \neq 0$, otherwise the proof is complete.

First of all, the function h is continuous.

This can be explained as follows. The function $x \mapsto f(x + \frac{1}{2})$ is the composition of two continuous functions. Hence, h is the difference of two continuous functions.

Furthermore

$$h(0) = f(0) - f(\tfrac{1}{2})$$

and

$$h(\tfrac{1}{2}) = f(\tfrac{1}{2}) - f(1) = f(\tfrac{1}{2}) - f(0) = -h(0).$$

So the numbers $h(0)$ and $h(\frac{1}{2})$ have an opposite sign. In this situation Exercise 3 implies that the function h has a zero.