6.11 Suppose that the function f is neither positive nor negative on the interval $[a,b]$. Then there exist $c_1, c_2 \in [a,b]$ such that $f(c_1) < 0$ and $f(c_2) > 0$. Say, $c_1 < c_2$.

Since the restriction of the function f to the compact interval $[c_1, c_2]$ is continuous and $f(c_1)f(c_2) < 0$, the Intermediate Value Theorem implies the existence of a $\tau \in [c_1, c_2]$ satisfying $f(\tau) = 0$. This is in contradiction with the data of the exercise.

7.1 As $\lim_{x\to 1}(x^2-1) = 0$ and $\lim_{x\to 1}\sqrt{x+1} = \sqrt{2}$, the Arithmetic Rules for limits of functions imply that

$$\lim_{x\to 1}(x^2-1)\sqrt{x+1} = \Big[\lim_{x\to 1}(x^2-1)\Big]\cdot\Big[\lim_{x\to 1}\sqrt{x+1}\Big] = 0\cdot\sqrt{2} = 0.$$

7.4 The sequence $\big(x_n\big)_{n=1}^{\infty}$ defined by

$$x_n = \frac{2}{(2n-1)\pi}$$

converges to 0, whereas

$$\begin{aligned}
\sin\frac{1}{x_1} &= \sin\frac{\pi}{2} = 1\\
\sin\frac{1}{x_2} &= \sin\frac{3\pi}{2} = -1\\
\sin\frac{1}{x_3} &= \sin\frac{5\pi}{2} = 1\\
\sin\frac{1}{x_4} &= \sin\frac{7\pi}{2} = -1\\
&\vdots
\end{aligned}$$

So the sequence of images $\big(\sin\frac{1}{x_n}\big)_{n=1}^{\infty}$ is in fact the alternating sequence. As this sequence diverges, the limit $\lim_{x\downarrow 0}\sin\frac{1}{x}$ doesn't exist.

7.5 The sequences $\big(x_n\big)_{n=1}^{\infty}$ and $\big(y_n\big)_{n=1}^{\infty}$ defined by

$$x_n = 1 + \frac{1}{n} \quad\text{and}\quad y_n = 1 - \frac{1}{n},$$

respectively, converge to 1, whereas

$$\lim_{n\to\infty} r(x_n) = \lim_{n\to\infty}\Big\{\big(1+\frac{1}{n}\big)^2 + 1\Big\} = 2$$

and

$$\lim_{n\to\infty} r(y_n) = \lim_{n\to\infty}\big(-1+\frac{1}{n} - 1\Big\} = -2.$$

Hence, the function r is not continuous at $x = 1$.

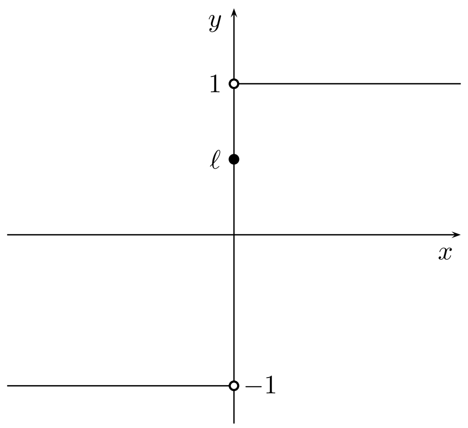
7.6 According to Example 9, the function h is not continuous if $\ell = 1$.

For the case $\ell \neq 1$, we choose the sequence $\big(x_n\big)_{n=1}^{\infty}$ defined by $x_n = \frac{1}{n}$. This sequence converges to 0, whereas

$$\lim_{n\to\infty} h(x_n) = \lim_{n\to\infty}\frac{\left|\frac{1}{n}\right|}{\frac{1}{n}} = 1 \neq h(0).$$

This proves that h is not continuous at 0.

The graph of the function h is presented below.



7.7 Let f be a function defined on an interval $(-\infty, a)$ and let ℓ be some number. We say that $f(x)$ converges to ℓ when $-x$ becomes large, if for every $\varepsilon > 0$ there exists a number H such that

$$|f(x) - \ell| < \varepsilon,$$

whenever $x < H$. This is denoted by $\lim_{x\to-\infty} f(x) = \ell$.

7.8 We will prove that $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = -1$.
Let $\varepsilon > 0$. Note that for $x > 0$

$$\left|\frac{\sqrt{x}-x}{\sqrt{x}+x} - (-1)\right| = \left|\frac{2\sqrt{x}}{\sqrt{x}+x}\right| = \frac{2}{1+\sqrt{x}} < \frac{2}{\sqrt{x}}.$$

Since, for $x > 0$, $\frac{2}{\sqrt{x}} < \varepsilon \iff x > \frac{4}{\varepsilon^2}$, we choose $H = \frac{4}{\varepsilon^2}$. Then for all $x > H$

$$\left|\frac{\sqrt{x}-x}{\sqrt{x}+x} - (-1)\right| < \frac{2}{\sqrt{x}} < \varepsilon.$$

This proves that $\lim_{x\to\infty} \frac{\sqrt{x}-x}{\sqrt{x}+x} = -1$. Hence $y = -1$ is a horizontal asymptote (at infinity) of the graph of the function g.

7.9 Let $\varepsilon > 0$. Note that for $x < -1$, $x^2 > -x$, so that

$$\left|\frac{\sqrt{1-x}}{x} - 0\right| = \frac{\sqrt{1-x}}{-x} = \sqrt{\frac{1}{x^2}(1-x)} = \sqrt{\frac{1}{x^2} - \frac{1}{x}} < \sqrt{-\frac{2}{x}}.$$

Since, for $x < -1$, $\sqrt{-\frac{2}{x}} < \varepsilon \iff x < -\frac{2}{\varepsilon^2}$, we choose $H = \min\{-1, -\frac{2}{\varepsilon^2}\}$. Then for all $x < H$

$$\left|\frac{\sqrt{1-x}}{x} - 0\right| < \sqrt{-\frac{2}{x}} < \varepsilon.$$

This proves that $\lim_{x\to-\infty} \frac{\sqrt{1-x}}{x} = 0$.

7.13 Let $(x_n)_{n=1}^{\infty}$ be a sequence in $I \setminus \{c\}$ which converges to c. Then $f(x_n) \to \ell$ as $n \to \infty$. Since $f(x_n) \geq 0$, Theorem 3.1 implies that $\ell \geq 0$.

7.15 Observe that for $x > 1$,

$$\begin{aligned}
\left|1 - x + \sqrt{x^2 - 2x + 3}\right| &= \left|\left[1 - x + \sqrt{x^2 - 2x + 3}\right] \frac{1 - x - \sqrt{x^2 - 2x + 3}}{1 - x - \sqrt{x^2 - 2x + 3}}\right| \\
&= \left|\frac{(1-x)^2 - (x^2 - 2x + 3)}{1 - x - \sqrt{x^2 - 2x + 3}}\right| = \left|\frac{1 - 2x + x^2 - x^2 + 2x - 3}{1 - x - \sqrt{x^2 - 2x + 3}}\right| \\
&= \left|\frac{-2}{1 - x - \sqrt{x^2 - 2x + 3}}\right| = \frac{2}{\sqrt{x^2 - 2x + 3} + x - 1} < \frac{2}{x - 1}.
\end{aligned}$$

Let $\varepsilon > 0$. As $\dfrac{2}{x-1} < \varepsilon \Longleftrightarrow x - 1 > \dfrac{2}{\varepsilon} \Longleftrightarrow x > 1 + \dfrac{2}{\varepsilon}$, we choose $H = \max\left\{1, 1 + \dfrac{2}{\varepsilon}\right\} = 1 + \dfrac{2}{\varepsilon}$. Then, for $x > H$,

$$\left|1 - x + \sqrt{x^2 - 2x + 3}\right| < \frac{2}{x - 1} < \varepsilon.$$

This proves that $\lim\limits_{x \to \infty} h(x) = 0$ or: the line $y = 0$ is a horizontal asymptote at infinity.
Note that for $x < 0$,

$$\begin{aligned}
|h(x) - (2 - 2x)| &= \ 1 - x + \sqrt{x^2 - 2x + 3} - 2 + 2x| = |x - 1 + \sqrt{x^2 - 2x + 3}| \\
&= \left|\left[x - 1 + \sqrt{x^2 - 2x + 3}\right] \frac{x - 1 - \sqrt{x^2 - 2x + 3}}{x - 1 - \sqrt{x^2 - 2x + 3}}\right| \\
&= \left|\frac{(x-1)^2 - (x^2 - 2x + 3)}{x - 1 - \sqrt{x^2 - 2x + 3}}\right| = \left|\frac{x^2 - 2x + 1 - x^2 + 2x - 3}{x - 1 - \sqrt{x^2 - 2x + 3}}\right| \\
&= \frac{2}{1 - x + \sqrt{x^2 - 2x + 3}} < \frac{2}{1 - x}.
\end{aligned}$$

Let $\varepsilon > 0$. As $\dfrac{2}{1-x} < \varepsilon \Longleftrightarrow 1 - x > \dfrac{2}{\varepsilon} \Longleftrightarrow x < 1 - \dfrac{2}{\varepsilon}$, we choose $L = \min\left\{0, 1 - \dfrac{2}{\varepsilon}\right\}$. Then, for $x < L$,

$$\left|1 - x + \sqrt{x^2 - 2x + 3}\right| < \frac{2}{1 - x} < \varepsilon.$$

This proves that $\lim\limits_{x \to \infty} \left[h(x) - (2 - 2x)\right] = 0$ or: the line $y = 2 - 2x$ is a linear asymptote at minus infinity.