6.12 Let h be the function on $[0,1]$ defined by

$$h(x) = f(x) - g(x) = x^2 - g(x).$$

As the function g is continuous, the function h is continuous too.

Furthermore,

$$h(0) = 0 - g(0) = -g(0) \leq 0$$

and $$h(1) = 1 - g(1) \geq 1 - 1 = 0.$$

If $h(0) = 0$ or $h(1) = 0$, then the equation $h(x) = 0 \Longleftrightarrow g(x) = x^2$ has a solution.

If $h(0) < 0$ and $h(1) > 0$, then the Intermediate Value Theorem applied to the function h implies that the equation $h(x) = 0 \Longleftrightarrow g(x) = x^2$ has a solution.

7.2 Consider the sequence

$$\left(1 + \frac{1}{n}\right)_{n=1}^{\infty}.$$

This sequence in D_f converges to 1 and $1 + \dfrac{1}{n} \neq 1$ for all $n \in \mathbb{N}$.

Now

$$f(1+\frac{1}{n}) = \frac{\sqrt{1+\frac{1}{n}}}{1-\sqrt{1+\frac{1}{n}}} = \frac{\sqrt{\frac{n+1}{n}}}{1-\sqrt{\frac{n+1}{n}}} = \frac{\sqrt{n+1}}{\sqrt{n}-\sqrt{n+1}} \cdot \frac{\sqrt{n}+\sqrt{n+1}}{\sqrt{n}+\sqrt{n+1}}$$

$$= -(\sqrt{n}\sqrt{n+1} + n + 1) = -(\sqrt{n^2+n} + n + 1) < -n.$$

Hence the sequence $\left(f(1+\frac{1}{n})\right)_{n=1}^{\infty}$ is not bounded below. So it is definitely not convergent (as you know each convergent sequence is bounded). According to the definition, the limit of $f(x)$ as $x \to 1$ doesn't exist.

7.3 Assume that $\lim\limits_{x\downarrow c} f(x) = \ell$ and $\lim\limits_{x\uparrow c} f(x) = \ell$.

Let $\left(x_n\right)_{n=1}^{\infty}$ be a sequence converging to c such that $x_n \neq c$ for all n.

We have to prove that $\lim\limits_{n\to\infty} f(x_n) = \ell$.

We will suppose that the sequence $\left(x_n\right)_{n=1}^{\infty}$ has an infinite number of terms larger than c and an infinite number of terms smaller than c. (Otherwise, the proof is simple.)

Since $x_n \to c$ as $n \to \infty$, the subsequence of $\left(x_n\right)_{n=1}^{\infty}$ consisting of the terms larger than c converges to c too. Because $\lim\limits_{x\downarrow c} f(x) = \ell$, a number N' exists such that

$$|f(x_n) - \ell| < \varepsilon,$$

whenever $n > N'$ **and** x_n is larger than c.

Similarly, a number N'' exists such that

$$|f(x_n) - \ell| < \varepsilon,$$

whenever $n > N'$ **and** x_n is smaller than c.

By consequence, for all $n > N = \max\{N', N''\}$,

$$|f(x_n) - \ell| < \varepsilon.$$

In other words: $\lim_{n\to\infty} f(x_n) = \ell$. As the sequence $\left(x_n\right)_{n=1}^{\infty}$ was arbitrarily chosen, this proves that $\lim_{x\to c} f(x) = \ell$.

7.10 Assume that the limit exists, say $\lim_{x\to\infty} \dfrac{3+x}{\sqrt{x}} = \ell$, where $\ell \geq 0$.
Then corresponding to $\varepsilon = 1$ a number H exists such that for all $x > H$

$$\left|\frac{3+x}{\sqrt{x}} - \ell\right| < 1 \Longrightarrow \frac{3+x}{\sqrt{x}} < 1 + \ell.$$

If however $x > (\ell+1)^2$, then

$$\frac{3+x}{\sqrt{x}} > \frac{x}{\sqrt{x}} = \sqrt{x} > 1 + \ell.$$

Since we have a contradiction, the limit doesn't exist.

7.12 For every $x > 0$,

$$|z(x) - (x+1)| = |\sqrt{1+x^2} + 1 - x - 1| = |\sqrt{1+x^2} - x| = \left|\left[\sqrt{1+x^2} - x\right]\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}+x}\right|$$
$$= \left|\frac{(1+x^2) - x^2}{\sqrt{1+x^2}+x}\right| = \frac{1}{\sqrt{1+x^2}+x} < \frac{1}{x}.$$

Let $\varepsilon > 0$. Take $H = \varepsilon^{-1}$. Then for $x > H$,

$$|z(x) - (x+1)| < \frac{1}{x} < \frac{1}{H} = \varepsilon.$$

This shows that $y = x + 1$ is a linear asymptote of the function z at infinity.

For every $x < 0$,

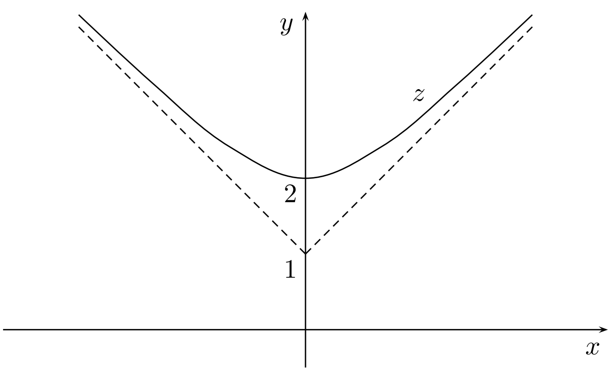
$$|z(x) - (1-x)| = |\sqrt{1+x^2} + 1 - 1 + x| = |\sqrt{1+x^2} + x| = \left|\left[\sqrt{1+x^2} + x\right]\frac{\sqrt{1+x^2}-x}{\sqrt{1+x^2}-x}\right|$$
$$= \left|\frac{(1+x^2) - x^2}{\sqrt{1+x^2}-x}\right| = \frac{1}{\sqrt{1+x^2}-x} < \frac{1}{-x}.$$

Let $\varepsilon > 0$. Take $H = -\varepsilon^{-1}$. Then for $x < H$,

$$|z(x) - (x+1)| < \frac{1}{-x} < \frac{1}{-H} = \varepsilon.$$

This shows that $y = 1 - x$ is a linear asymptote of the function z at minus infinity.

The graph of the function z is represented below.



7.14 The sequence $\big(x_n\big)_{n=1}^{\infty}$ with $x_n = \dfrac{1}{n}$ converges to 0, whereas

$$g(x_n) = \frac{\sqrt{1+\dfrac{1}{n}}-1}{\dfrac{1}{n}} = n\sqrt{1+\frac{1}{n}} - n = \sqrt{n^2+n} - n$$
$$= \left[\sqrt{n^2+n}-n\right]\frac{\sqrt{n^2+n}+n}{\sqrt{n^2+n}+n} = \frac{n^2+n-n^2}{\sqrt{n^2+n}+n} = \frac{1}{\sqrt{1+\dfrac{1}{n}}+1}.$$

Hence, $\lim\limits_{n\to\infty} g(x_n) = \frac{1}{2}$.

The sequence $\big(y_n\big)_{n=1}^{\infty}$ with $y_n = -\dfrac{1}{n}$ converges to 0, whereas

$$g(y_n) = \frac{\sqrt{1-\dfrac{1}{n}}-1}{\dfrac{1}{n}} = n\sqrt{1-\frac{1}{n}} - n = \sqrt{n^2-n} - n$$
$$= \left[\sqrt{n^2-n}-n\right]\frac{\sqrt{n^2-n}+n}{\sqrt{n^2-n}+n} = \frac{n^2-n-n^2}{\sqrt{n^2-n}+n} = -\frac{1}{\sqrt{1-\dfrac{1}{n}}+1}.$$

Hence, $\lim\limits_{n\to\infty} g(y_n) = -\frac{1}{2}$.

The foregoing implies that the limit $\lim\limits_{x\to 0} g(x)$ doesn't exist.

7.18 (a) Because $\lim\limits_{x\to-\infty} f(x) = 0$, for $\varepsilon = f(0) > 0$ an $H_1 \in \mathbb{R}$ exists such that

$$f(x) < \varepsilon = f(0),$$

whenever $x < H_1$. This however means that $0 > H_1$.

Because $\lim\limits_{x\to\infty} f(x) = 0$, an $H_2 > H_1 \in \mathbb{R}$ exists such that

$$f(x) < \varepsilon = f(0),$$

whenever $x > H_2$. This however means that $0 < H_2$.

(b) The function f restricted to the compact interval $[H_1, H_2]$ is continuous. According to the Theorem of Weierstrass, the function f has a maximum $f(d)$ on the interval $[H_1, H_2]$.

Then, however, $f(d)$ is also the maximum of the function f on $\mathbb{R}$. This can be seen as follows:

$$f(d) \begin{cases} \geq f(0) > f(x) & \text{if} \quad x < H_1 \\ \geq f(x) & \text{if} \quad x \in [H_1, H_2] \\ \geq f(0) > f(x) & \text{if} \quad x > H_2. \end{cases}$$