9.11 (a) The set V is not a linear subspace of $\mathbb{R}^3$ because it is not closed with respect to the scalar multiplication.

For example: $\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \in V$, but $2\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \notin V$.

(b) The set W is not a linear subspace of $\mathbb{R}^3$ because it is not closed with respect to the scalar multiplication.

For example: $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \in W$, but $2\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \notin W$.

9.12 (a) The set

$$U = \{A \in \mathbb{M}_{2\times 2} |\ \det A = 0\}$$

is not a linear subspace of $\mathbb{M}_{2\times 2}$, because the set is not closed with respect to the addition.

Consider for example the matrices $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$. Then $\det A = \det B = 0$, so $A, B \in U$.

However $A + B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $\det(A+B) = 1$. So $A + B \notin U$.

(b) The set

$$V = \{x \mapsto a_1 x + a_2 x^2 |\ a_1 \text{ and } a_2 \text{ whole numbers}\}$$

is not a linear subspace of $\mathbb{P}^3$, because the set is not closed with respect to the scalar multiplication.

Consider for example the polynomial $p\colon x \to x + x^2$. Then $p \in V$. However, since $\frac{1}{2}p(x) = \frac{1}{2}x + \frac{1}{2}x^2$, $\frac{1}{2}p \notin V$.

(c) The set

$$W = \{f \in \mathbb{F} |\ f(0) = 2\}$$

is not a linear subspace of $\mathbb{F}$, because the set is not closed with respect to the scalar multiplication.

Consider for example the function f defined by $f(x) = 2$ for $x \in \mathbb{R}$. Then $f \in W$. However $\frac{1}{2}f \notin W$ because $\frac{1}{2}f(0) = 1 \neq 2$.

9.13 (a) We have to check whether an $\underline{x} \in \mathbb{R}^4$ exists such that $T(\underline{x}) = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}$.

So in fact we have to check the solvability of the system $A\underline{x} = \underline{b}$, where

$$A = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix} \qquad \text{and} \qquad \underline{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}.$$

(Partial) reduction of the augmented coefficient matrix of the system leads to:

$$\begin{bmatrix} 4 & 1 & -2 & -3 & 1 \\ 2 & 1 & 1 & -4 & 3 \\ 6 & 0 & -9 & 9 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & -1 & -4 & 5 & -5 \\ 0 & -3 & -12 & 21 & -9 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & -2 & \frac{3}{2} \\ 0 & 1 & 4 & -5 & 5 \\ 0 & -3 & -12 & 21 & -9 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -1\frac{1}{2} & \frac{1}{2} & -1 \\ 0 & 1 & 4 & -5 & 5 \\ 0 & 0 & 0 & 6 & 6 \end{bmatrix}.$$

Hence, the system has a solution which implies that $\underline{b} \in \text{Im}(T)$.

(b) We have to check whether $T(\underline{e}_4) = \underline{0}$. Well, $T(\underline{e}_4) = \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix}$. By consequence, $\underline{e}_4 \notin \text{Ker}(T)$.

9.15 (a) Let $p: x \to 1 + x$. Then $p \in \mathbb{P}_2$ and $T(p) = q$, where $q(x) = x + x^2$. So q is the image of the polynomial p, that is: q is contained in the range of the mapping T.

(b) If $p \in \text{Ker}(T)$, then $T(p) = 0$, where 0 denotes the null function.

So $xp(x) = 0$ for all $x \in \mathbb{R}$. This is possible only if $p(x) = 0$ for all $x \in \mathbb{R} \setminus \{0\}$. Since p is a continuous function, this implies that $p(x) = 0$ for all $x \in \mathbb{R}$. Hence, $\text{Ker}(T) = \{0\}$.

9.17 We have to check whether the system $A\underline{x} = \underline{b}$ is solvable, where

$$A = \begin{bmatrix} 0 & 1 \\ -2 & 3 \\ 2 & -1 \end{bmatrix} \qquad \text{and} \qquad \underline{b} = \begin{bmatrix} 0 \\ 4 \\ 5 \end{bmatrix}.$$

In order to do so we reduce the augmented coefficient matrix of the system:

$$\begin{bmatrix} 0 & 1 & 0 \\ -2 & 3 & 4 \\ 2 & -1 & 5 \end{bmatrix} \to \begin{bmatrix} 2 & -1 & 5 \\ -2 & 3 & 4 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & -\frac{1}{2} & \frac{5}{2} \\ 0 & 2 & 9 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{19}{4} \\ 0 & 1 & \frac{9}{2} \\ 0 & 0 & -\frac{9}{2} \end{bmatrix}.$$

Since the system is not solvable, the vector $\underline{b}$ cannot be written as a linear combination of the vectors corresponding with the columns of the matrix A.

9.18 We are looking for numbers c_1, c_2 and c_3 such that $p = c_1p_1 + c_2p_2 + c_3p_3$.

Then, for all $x \in \mathbb{R}$,

$$p(x) = c_1p_1(x) + c_2p_2(x) + c_3p_3(x)$$

or

$$\begin{aligned} -9 - 7x - 15x^2 &= c_1(2 + x + 4x^2) + c_2(1 - x + 3x^2) + c_3(3 + 2x + 5x^2) \\ &= 2c_1 + c_2 + 3c_3 + (c_1 - c_2 + 2c_3)x + (4c_1 + 3c_2 + 5c_3)x^3. \end{aligned}$$

This is possible only if the corresponding coefficients are equal. So we have to solve the system

$$\begin{cases} 2c_1 + c_2 + 3c_3 = -9 \\ c_1 - c_2 + 2c_3 = -7 \\ 4c_1 + 3c_2 + 5c_3 = -15. \end{cases}$$

Reduction of the augmented coefficient matrix of this system leads to

$$\begin{bmatrix} 2 & 1 & 3 & -9 \\ 1 & -1 & 2 & -7 \\ 4 & 3 & 5 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -7 \\ 2 & 1 & 3 & -9 \\ 4 & 3 & 5 & -15 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 3 & -1 & 5 \\ 0 & 7 & -3 & 13 \end{bmatrix} \to \begin{bmatrix} 1 & -1 & 2 & -7 \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 7 & -3 & 13 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{5}{3} & -\frac{16}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & -\frac{2}{3} & \frac{4}{3} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{5}{3} & -\frac{16}{3} \\ 0 & 1 & -\frac{1}{3} & \frac{5}{3} \\ 0 & 0 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & -2 \end{bmatrix}.$$

So $c_1 = -2$, $c_2 = 1$ and $c_3 = -2$, that is: $p = -2p_1 + p_2 - 2p_3$.

9.19 We are looking for numbers c_1, c_2 and c_3 such that

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = c_1 A + c_2 B + c_3 C \Longleftrightarrow \begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = \begin{bmatrix} 4c_1 + c_2 & -c_2 + 2c_3 \\ -2c_1 + 2c_2 + c_3 & -2c_1 + 3c_2 + 4c_3 \end{bmatrix}.$$

So we have to solve the system

$$\begin{cases} 4c_1 + c_2 = 6 \\ -c_2 + 2c_3 = -8 \\ -2c_1 + 2c_2 + c_3 = -1 \\ -2c_1 + 3c_2 + 4c_3 = -8. \end{cases}$$

Reduction of the augmented coefficient matrix of this system leads to

$$\begin{bmatrix} 4 & 1 & 0 & 6 \\ 0 & -1 & 2 & -8 \\ -2 & 2 & 1 & -1 \\ -2 & 3 & 4 & -8 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & -1 & 2 & -8 \\ 0 & 2\frac{1}{2} & 1 & 2 \\ 0 & 3\frac{1}{2} & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{4} & 0 & \frac{3}{2} \\ 0 & 1 & -2 & 8 \\ 0 & 2\frac{1}{2} & 1 & 2 \\ 0 & 3\frac{1}{2} & 4 & -5 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 6 & -18 \\ 0 & 0 & 11 & -33 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & \frac{1}{2} & -\frac{1}{2} \\ 0 & 1 & -2 & 8 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

So $c_1 = 1, c_2 = 2$ and $c_3 = -3$, that is:

$$\begin{bmatrix} 6 & -8 \\ -1 & -8 \end{bmatrix} = A + 2B - 3C.$$

The zero matrix is also a linear combination of the matrices A, B and C: take all the weighs equal to 0.

9.22 Let $\underline{v} \in S$. As $\underline{v}_i$ is a linear combination of the vectors $\underline{w}_1, \ldots, \underline{w}_n$, $\underline{v}_i \in \text{span}\{\underline{w}_1, \ldots, \underline{w}_n\} = T$. So $\underline{v}$ is a linear combination of vectors in T. Then $\underline{v} \in T$. As $\underline{v}$ was arbitrarily chosen, this implies that $S \subset T$.

In a similar way one proves that $T \subset S$.

10.1 (a) Obviously, $\underline{v}_2 = -5\,\underline{v}_1$. So by Theorem 1, the two vectors are linearly dependent.

(b) According to Theorem 2, three vectors in $\mathbb{R}^2$ are linearly dependent.

(c) Obviously, $p_2 = 2p_1$. So by Theorem 1, the two polynomials are linearly dependent.

(d) Obviously, $B = -A$. So by Theorem 1, the two matrices are linearly dependent.

10.2 (a) Since the one vector is not a multiple of the other one, the vectors are linearly independent.

(b) We have to investigate whether numbers c_1, c_2 and c_3 (not all of them equal to zero) exist such that $c_1\,\underline{v}_1 + c_2\,\underline{v}_2 + c_3\,\underline{v}_3 = \underline{0}$.

Equivalently, we have to investigate, whether the system $A\underline{x} = \underline{0}$ has a non-trivial solution, where

$$A = \begin{bmatrix} -3 & 5 & 1 \\ 0 & -1 & 1 \\ 4 & 2 & 3 \end{bmatrix}.$$

Well, since

$$\det A = -3 \cdot \det \begin{bmatrix} -1 & 1 \\ 2 & 3 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 5 & 1 \\ -1 & 1 \end{bmatrix} = -3 \cdot (-5) + 4 \cdot 6 = 39 \neq 0,$$

the matrix A is invertible. So, according to Theorem 7.6, the system $A\underline{x} = \underline{0}$ only has the trivial solution.

Hence, $c_1 = c_2 = c_3 = 0$, which means that the vectors are linearly independent.

10.3 We have to investigate whether numbers c_1, c_2 and c_3 (not all of them equal to zero) exist such that $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$, where 0 denotes the null function.

Now $c_1 p_1 + c_2 p_2 + c_3 p_3 = 0$ if and only if, for all $x \in \mathbb{R}$,

$$c_1(2 - x + 4x^2) + c_2(3 + 6x + 2x^2) + c_3(2 + 10x - 4x^2) = 0.$$

This means that, for all $x \in \mathbb{R}$,

$$2c_1 + 3c_2 + 2c_3 + (-c_1 + 6c_2 + 10c_3)x + (4c_1 + 2c_2 - 4c_3)x^2 = 0.$$

This is possible only if all coefficients are equal to zero (the number of zeros of a polynomial of degree two is at most equal to two). So the constants c_1, c_2 and c_3 satisfy

$$\begin{cases} 2c_1 + 3c_2 + 2c_3 = 0 \\ -c_1 + 6c_2 + 10c_3 = 0 \\ 4c_1 + 2c_2 - 4c_3 = 0. \end{cases}$$

(Partial) reduction of the coefficient matrix of this system leads to

$$\begin{bmatrix} 2 & 3 & 2 \\ -1 & 6 & 10 \\ 4 & 2 & -4 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & -10 \\ 0 & 15 & 22 \\ 0 & 26 & 36 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & -10 \\ 0 & 1 & \frac{22}{15} \\ 0 & 0 & 36 - \frac{26 \times 22}{15} \end{bmatrix}.$$

Hence, $c_1 = c_2 = c_3 = 0$. According to the definition, the polynomials p_1, p_2 and p_3 are linearly independent.

10.4 First we will prove that the vectors $\underline{v}_1, \ldots, \underline{v}_{n-1}$ are linearly independent.

Assume that $c_1 \underline{v}_1 + \cdots + c_{n-1} \underline{v}_{n-1} = \underline{0}$, for some numbers $c_1, \ldots, c_{n-1}$.

Then $c_1 \underline{v}_1 + \cdots + c_{n-1} \underline{v}_{n-1} + 0 \, \underline{v}_n = \underline{0}$.

Since the vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent this implies that $c_1 = c_2 = \ldots = c_{n-1} = 0$. So the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_{n-1}$ are linearly independent

Next we will prove that the vectors $\underline{v}_1, \ldots, \underline{v}_{n-1}$ do not span V by showing that $\underline{v}_n \notin \operatorname{span}\{\underline{v}_1, \ldots, \underline{v}_{n-1}\}$.

Suppose that the vector $\underline{v}_n$ is in this span. Then constants $c_1, \ldots, c_{n-1}$ exist such that $\underline{v}_n = c_1 \underline{v}_1 + \cdots + c_{n-1} \underline{v}_{n-1}$. Then, however, Theorem 1 implies that the vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly dependent. This is a contradiction.