9.9 (b) We will show, by means of a counterexample, that the mapping $T$ is not linear.

Consider the polynomial $p$, where $p(x) = 1$ for all $x \in \mathbb{R}$.

Then $T(p)$ is the polynomial

$$x \mapsto 2 + x + x^2.$$

So $T(2p)$ is the polynomial

$$x \mapsto 3 + x + x^2,$$

while $2T(p)$ is the polynomial

$$x \mapsto 4 + 2x + 2x^2.$$

Hence, $T(2p) \neq 2T(p)$.

9.10 As $\underline{0} \in \text{Col}(A)$, $\text{Col}(A)$ is nonempty.

Assume that $\underline{b}$ and $\underline{b}'$ are contained in $\text{Col}(A)$ and that $c$ is a real number. Then vectors $\underline{s}$ and $\underline{s}'$ (of proper dimension) exist such that $A\underline{s} = \underline{b}$ and $A\underline{s}' = \underline{b}'$.

Then

$$A(\underline{s} + \underline{s}') = A\underline{s} + A\underline{s}' = \underline{b} + \underline{b}'.$$

This implies that $\underline{b} + \underline{b}' \in \text{Col}(A)$.

In a similar way one shows that $c\,\underline{b} \in \text{Col}(A)$.

9.12 (d) The set

$$Z = \{f \in \mathbb{D} |\ f' + 2f = 0\}$$

is a linear subspace of $\mathbb{D}$.

First of all note that $Z$ is a non-empty set, because $Z$ contains the null function (which is differentiable).

In order to prove that the set $Z$ is closed with respect to the addition, we assume that $f, g \in Z$.

Then the functions $f$ and $g$ are differentiable and $f' + 2f = 0$ and $g' + 2g = 0$. Hence the function $f + g$ is differentiable and for all $x \in \mathbb{R}$

$$(f + g)'(x) + 2(f + g)(x) = f'(x) + 2f(x) + g'(x) + 2g(x) = 0.$$

So $(f + g)' + 2(f + g) = 0$, that is: $f + g \in Z$.

In order to prove that the set $Z$ is closed with respect to the scalar multiplication, we assume that $f \in Z$ and $c \in \mathbb{R}$.

Then the function $f$ is differentiable and $f' + 2f = 0$. Hence the function $cf$ is differentiable and for all $x \in \mathbb{R}$

$$(cf)'(x) + 2(cf)(x) = cf'(x) + 2cf(x) = 0.$$

So $(cf)' + 2(cf) = 0$, that is: $cf \in Z$.

9.14 Note that $\text{Ker}(D) = \{p \in \mathbb{P}^3 |\ D(p) = 0\}$. Here 0 denotes the null function.

Because $D(p) = p'$, $D(p) = 0$ if and only if $p' = 0$.

The only polynomials whose derivative is the null function are the constant functions.

So the set of constant functions is the kernel of $D$.

9.16 Assume that the mapping $T$ is one-to-one.

If $\underline{v} \in \text{Ker}(T)$, then $T(\underline{v}) = \underline{0} = T(\underline{0})$. Because the mapping is one-to-one, this implies that $\underline{v} = \underline{0}$.

Hence, $\text{Ker}(T) = \{\underline{0}\}$.

Assume that $\text{Ker}(T) = \{\underline{0}\}$.

Suppose that $T(\underline{v}_1) = T(\underline{v}_2)$ for two vectors $\underline{v}_1, \underline{v}_2 \in V$. Since the mapping is linear, this implies that

$$T(\underline{v}_1 - \underline{v}_2) = T(\underline{v}_1) - T(\underline{v}_2) = \underline{0}.$$

So $\underline{v}_1 - \underline{v}_2 \in \text{Ker}(T)$. Hence, $\underline{v}_1 - \underline{v}_2 = \underline{0}$, or $\underline{v}_1 = \underline{v}_2$. This means that the mapping $T$ is one-to-one.

9.20 Each element in the set $W$ can be written as

$$\begin{bmatrix} a-3b \\ b-a \\ a \\ b \end{bmatrix} = a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

By consequence

$$W = \left\{ a \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \in \mathbb{R}^4 \Big|\ a, b \in \mathbb{R} \right\} = \text{span}\left\{ \begin{bmatrix} 1 \\ -1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 1 \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Hence, according to Theorem 5, $W$ is a linear subspace.

9.23 Assume that $\underline{v}_n = c_1\,\underline{v}_1 + \cdots + c_{n-1}\,\underline{v}_{n-1}$ for some numbers $c_1, \ldots, c_{n-1}$.

We will show that $V = \text{span}\{\underline{v}_1, \ldots, \underline{v}_{n-1}\}$.

First we will prove the inclusion $\subset$.

Let $\underline{v} \in V$. Then, for some numbers $d_1, \ldots, d_n$,

$$\begin{aligned} \underline{v} &= d_1\,\underline{v}_1 + \cdots + d_n\,\underline{v}_n = d_1\,\underline{v}_1 + \cdots + d_{n-1}\,\underline{v}_{n-1} + d_n[c_1\,\underline{v}_1 + \cdots + c_{n-1}\,\underline{v}_{n-1}] \\ &= d_1\,\underline{v}_1 + \cdots + d_{n-1}\,\underline{v}_{n-1} + d_n c_1\,\underline{v}_1 + \cdots + d_n c_{n-1}\,\underline{v}_{n-1} \\ &= (d_1 + d_n c_1)\,\underline{v}_1 + \cdots + (d_{n-1} + d_n c_{n-1})\,\underline{v}_{n-1}. \end{aligned}$$

Hence, $\underline{v} \in \text{span}\{\underline{v}_1, \ldots, \underline{v}_{n-1}\}$. By consequence, $V \subset \text{span}\{\underline{v}_1, \ldots, \underline{v}_{n-1}\}$.

Finally we will prove the inclusion $\supset$.

Since $V$ is a vector space and $\underline{v}_1, \ldots, \underline{v}_{n-1} \in V$, also $\text{span}\{\underline{v}_1, \ldots, \underline{v}_{n-1}\} \subset V$.

9.24 Obviously, $T(\underline{v}_1), \ldots, T(\underline{v}_n) \in W$. Because $W$ is a vector space, $\text{span}\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\} \subset W$. So it is sufficient to show that $W \subset \text{span}\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$.

Let $\underline{w} \in W$. Because the mapping $T$ is surjective, there exists a vector $\underline{v} \in V$ with $T(\underline{v}) = \underline{w}$.

Because the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ span the space $V$, there exist numbers $c_1, c_2, \ldots, c_n$ such that

$$\underline{v} = c_1\,\underline{v}_1 + \cdots + c_n\,\underline{v}_n.$$

The fact that the mapping $T$ is linear implies that

$$\underline{w} = T(\underline{v}) = T(c_1\,\underline{v}_1 + \cdots + c_n\,\underline{v}_n) = c_1 T(\underline{v}_1) + \cdots + c_n T(\underline{v}_n).$$

Hence, $\underline{w} \in \text{span}\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$.

Since the vector $\underline{w}$ was arbitrarily chosen, we may conclude that $W \subset \text{span}\{T(\underline{v}_1), \ldots, T(\underline{v}_n)\}$.

10.5 Since the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_n$ are linearly dependent, according to Theorem 1, at least one of these vectors is a linear combination of the other ones. Say $\underline{v}_n$ is a linear combination of the other vectors:

$$\underline{v}_n = c_1\,\underline{v}_1 + \cdots + c_{n-1}\,\underline{v}_{n-1},$$

for certain numbers $c_1, \ldots, c_{n-1}$. Then

$$\underline{v}_n = c_1\,\underline{v}_1 + \cdots + c_{n-1}\,\underline{v}_{n-1} + 0\,\underline{v},$$

which means that $\underline{v}_n$ is a linear combination of the vectors $\underline{v}_1, \ldots, \underline{v}_{n-1}, \underline{v}$. By Theorem 1, the vectors $\underline{v}, \underline{v}_1, \ldots, \underline{v}_n$ are linearly dependent.

10.6 Suppose that $c_1\,\underline{u} + c_2\,\underline{v} + c_3\,\underline{w} = \underline{0}$ for some numbers $c_1, c_2$ en $c_3$.

This is possible only if $c_3 = 0$. For suppose that $c_3 \neq 0$. Then

$$\underline{w} = -\frac{c_1}{c_3}\underline{u} - \frac{c_2}{c_3}\underline{v}.$$

This however means that $\underline{w} \in \text{span}\{\underline{u}, \underline{v}\}$ which is in contradiction with the data of the exercise.

Now the fact that $c_3 = 0$ implies that

$$c_1\,\underline{u} + c_2\,\underline{v} = \underline{0}.$$

Because the vectors $\underline{u}$ en $\underline{v}$ are linearly independent this implies that $c_1 = c_2 = 0$.

As a result $c_1 = c_2 = c_3 = 0$, which proves that the vectors $\underline{u}, \underline{v}$ and $\underline{w}$ are linearly independent.