10.7 The matrices $D_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$, $D_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ and $D_3 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ in $\mathbb{D}_{3\times 3}$ form a basis of $\mathbb{D}_{3\times 3}$.

In order to prove that these matrices are linearly independent, assume that, for some numbers c_1, c_2 and c_3, $c_1 D_1 + c_2 D_2 + c_3 D_3 = O$. Then

$$\begin{bmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longleftrightarrow c_1 = c_2 = c_3 = 0.$$

So the matrices D_1, D_2 and D_3 are linearly independent.

Finally, we prove that $\mathbb{D}_{3\times 3} = \operatorname{span}\{D_1, D_2, D_3\}$.

Let $D \in \mathbb{D}_{3\times 3}$, so

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix},$$

for some numbers d_1, d_2 and d_3. Then $D = d_1 D_1 + d_2 D_2 + d_3 D_3$, that is: $\mathbb{D}_{3\times 3} \subset \operatorname{span}\{D_1, D_2, D_3\}$. Since the matrices D_1, D_2 and D_3 are diagonal matrices and $\mathbb{D}_{3\times 3}$ is a vector space, also $\operatorname{span}\{D_1, D_2, D_3\} \subset \mathbb{D}_{3\times 3}$.

10.8 (a) According to Theorem 2, three vectors in the space $\mathbb{R}^2$ are linearly dependent.

(b) The two vectors do not span the space $\mathbb{R}^3$: the vector $\underline{e}_3 \in \mathbb{R}^3$ is not contained in $\operatorname{span}\{\underline{v}_1, \underline{v}_2\}$. This can be seen as follows. In order to find numbers c_1 and c_2 satisfying $\underline{e}_3 = c_1 \underline{v}_1 + c_2 \underline{v}_2$ we reduce the following augmented coefficient matrix

$$\begin{bmatrix} -1 & 6 & 0 \\ 3 & 1 & 0 \\ 2 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -6 & 0 \\ 0 & 19 & 0 \\ 0 & 13 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So such numbers c_1 and c_2 do not exist.

(c) Lets denote the two polynomials by p_1 and p_2, respectively. These polynomials do not span the space $\mathbb{P}_2$, because the polynomial $p_3 \colon x \to 1$ is not contained in $\operatorname{span}\{p_1, p_2\}$.

If $p_3 = c_1 p_1 + c_2 p_2$, for some numbers c_1 and c_2, then these numbers should satisfy

$$\begin{cases} c_1 - c_2 = 1 \\ c_1 + c_2 = 0 \\ \phantom{c_1 + {}} c_1 = 0. \end{cases}$$

Obviously, this system is inconsistent.

(d) The five matrices are linearly dependent.

Suppose that $\sum_{i=1}^{5} c_i M_i = 0$, for some numbers $c_1, \ldots, c_5$. Then these numbers are a solution of a homogeneous system of linear equations with five variables and four equations. Obviously, at least one of these variables is a free variable. So the system has an infinite number of solutions. Hence, the system has a non-trivial solution, which means that the matrices are linearly dependent.

10.9 (a) Let

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 0 \end{bmatrix}.$$

Since $\det A = -3 \neq 0$, the matrix A is invertible. Hence, the system $A\underline{x} = \underline{0}$ has only the trivial solution. This implies that the two vectors (corresponding to the columns of the matrix A) are linearly independent.

Furthermore, the system $A\underline{x} = \underline{b}$ is solvable for any $\underline{b} \in \mathbb{R}^2$. Hence, the two vectors span the space $\mathbb{R}^2$.

(b) This can be solved as in part (a). In this case $\det A = -32 + 7 = -25 \neq 0$.

(c) The two vectors are not linearly independent:

$$5 \cdot \begin{bmatrix} 0 \\ 0 \end{bmatrix} + 0 \cdot \begin{bmatrix} 1 \\ 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

(d) The two vectors are not linearly independent:

$$\begin{bmatrix} -4 \\ -12 \end{bmatrix} = -\tfrac{4}{3} \begin{bmatrix} 3 \\ 9 \end{bmatrix}.$$

10.10 Let's denote the polynomials by p_1, p_2 and p_3, respectively. In order to show that the polynomials form a basis of the space $\mathbb{P}_2$, we will show that they are linearly independent and that they span the space $\mathbb{P}_2$.

In order to show that the polynomials p_1, p_2 en p_3 are independent, we suppose that $c_1p_1+c_2p_2+c_3p_3 = 0$, for some numbers c_1, c_2 en c_3. Then, for all $x \in \mathbb{R}$,

$$c_1p_1(x) + c_2p_2(x) + c_3p_3(x) = 0 \Longrightarrow c_1(1 + x + x^2) + c_2(x + x^2) + c_3x^2 = 0$$

$$\Longrightarrow c_1 + (c_1 + c_2)x + (c_1 + c_2 + c_3)x^2 = 0.$$

Because a polynomial is the the zero polynomial only if all coefficients are equal to zero, the numbers c_1, c_2 and c_3 satisfy

$$\begin{cases} c_1 & & & = & 0 \\ c_1 & + & c_2 & & = & 0 \\ c_1 & + & c_2 & + & c_3 & = & 0. \end{cases}$$

By consequence, $c_1 = c_2 = c_3 = 0$, which implies that the polynomials p_1, p_2 and p_3 are linearly independent.

We will show that the polynomials p_1, p_2 and p_3 span the space $\mathbb{P}_2$.

Obviously, $p_1, p_2, p_3 \in \mathbb{P}_2$. Let $p \in \mathbb{P}_2$, say

$$p(x) = a + bx + cx^2, \qquad (x \in \mathbb{R})$$

for some numbers a, b and c.

In order to check whether there exist numbers c_1, c_2 and c_3 such that $p = c_1p_1 + c_2p_2 + c_3p_3$, we have to investigate the solvability of the system

$$\begin{cases} c_1 & = a \\ c_1 + c_2 & = b \\ c_1 + c_2 + c_3 & = c. \end{cases}$$

Well, because the coefficient matrix of this system is invertible, the system is solvable for all $a, b, c \in \mathbb{R}$. Hence, $\mathbb{P}_2 = \text{span}\{p_1, p_2, p_3\}$.

10.12 (a) Reduction of the coefficient matrix of the system leads to

$$\begin{bmatrix} 1 & 1 & -1 \\ -2 & -1 & 2 \\ -1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So x_3 is a free variable and each solution of the system is of the form

$$\begin{bmatrix} x_3 \\ 0 \\ x_3 \end{bmatrix} = x_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

where $x_3 \in \mathbb{R}$. So the solution set is $\text{span}\left\{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}\right\}$. Hence, the vector $\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$ forms a basis of the solution set.

(b) Reduction of the coefficient matrix of the system leads to

$$\begin{bmatrix} 1 & -3 & 1 \\ 2 & -6 & 2 \\ 3 & -9 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So x_2 and x_3 are free variables and each solution of the system is of the form

$$\begin{bmatrix} 3x_2 - x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix},$$

where $x_2, x_3 \in \mathbb{R}$. So the solution set is $\text{span}\left\{\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}\right\}$. Hence, the vectors $\begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ form a basis of the solution set (they are clearly independent).

10.14 (a) Reduction of the matrix leads to the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the columns 1 and 3 of this reduced matrix contain the leading ones, the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$$

that correspond to the columns 1 and 3 of the matrix A form a basis of $\text{Col}(A)$.

(b) Reduction of the matrix leads to the matrix

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the columns 1 and 2 of this reduced matrix contain the leading ones, the vectors

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

that correspond to the columns 1 and 2 of the matrix A form a basis of $\text{Col}(A)$.

(c) Obviously, reduction of the matrix leads to a matrix where each column contains a leading one. Hence, all the columns of the matrix A form a basis of $\text{Col}(A)$.

10.15 (a) Reduction of the matrix leads to

$$\begin{bmatrix} 1 & -1 & 3 \\ 5 & -4 & -4 \\ 7 & -6 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 1 & -19 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 3 \\ 0 & 1 & -19 \\ 0 & 0 & 0 \end{bmatrix}.$$

Since the columns 1 and 2 of this matrix contain the leading ones, the vectors

$$\begin{bmatrix} 1 \\ 5 \\ 7 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ -4 \\ -6 \end{bmatrix},$$

that correspond to the columns 1 and 2 of the matrix A, form a basis of $\text{Col}(A)$.

(b) Reduction of the matrix leads to

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 1 & 1 & \frac{4}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the columns 1 and 2 of this matrix contain the leading ones, the vectors

$$\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 4 \\ 1 \\ 3 \end{bmatrix},$$

that correspond to the columns 1 and 2 of the matrix A, form a basis of $\text{Col}(A)$.

10.17 (a) Obviously, the mapping T is the left-multiplication by the (standard) matrix

$$A = \begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix}.$$

Since $\text{Ker}(T) = \text{Null}(A)$, we reduce the matrix A

$$\begin{bmatrix} 4 & 1 & -2 & -3 \\ 2 & 1 & 1 & -4 \\ 6 & 0 & -9 & 9 \end{bmatrix} \rightarrow \begin{bmatrix} 6 & 0 & -9 & 9 \\ 2 & 1 & 1 & -4 \\ 4 & 1 & -2 & -3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} & \frac{3}{2} \\ 0 & 1 & 4 & -7 \\ 0 & 1 & 4 & -9 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & -\frac{3}{2} & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

So x_3 is a free variable and each vector in $\text{Null}(A)$ has the form

$$x_3 \begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix},$$

where $x_3 \in \mathbb{R}$. So the vector

$$\begin{bmatrix} \frac{3}{2} \\ -4 \\ 1 \\ 0 \end{bmatrix}$$

forms a basis of the kernel of T.

(b) Note that $\mathrm{Im}(T) = \mathrm{Col}(A)$. Since the columns 1, 2 and 4 contain the leading ones in the reduced matrix of A, the vectors

$$\begin{bmatrix} 4 \\ 2 \\ 6 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ -4 \\ 9 \end{bmatrix},$$

which correspond to the columns 1, 2 and 3 of the matrix A, form a basis of $\mathrm{Im}(T)$.