10.11 Let $\underline{v} \in V$ be a vector with $\underline{v} \notin \text{span}\{\underline{v}_1, \ldots, \underline{v}_n\}$ and suppose that

$$c_0\, \underline{v} + c_1\, \underline{v}_1 + \cdots + c_n\, \underline{v}_n = \underline{0},$$

for some numbers $c_0, \ldots c_n$. Assume that $c_0 \neq 0$. Then

$$\underline{v} = -\frac{c_1}{c_0}\, \underline{v}_1 + \cdots + \left(-\frac{c_n}{c_0}\right) \underline{v}_n.$$

This is in contradiction with the fact that $\underline{v} \notin \text{span}\{\underline{v}_1, \ldots, \underline{v}_n\}$.

So $c_0 = 0$. Then, however, $c_1\, \underline{v}_1 + \cdots + c_n\, \underline{v}_n = \underline{0}$. Because the vectors $\underline{v}_1, \ldots, \underline{v}_n$ are linearly independent, this implies that $c_1 = \ldots = c_n = 0$. So the vectors $\underline{v}, \underline{v}_1, \ldots, \underline{v}_n$ are linearly independent.

10.13 (a) The set of solutions of the 'system' $3x - 2y + 5z = 0$ is

$$\left\{ \begin{bmatrix} \frac{2}{3}y - \frac{5}{3}z \\ y \\ z \end{bmatrix} \mid y, z \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \mid y, z \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix} \right\}.$$

Because the two vectors $\begin{bmatrix} \frac{2}{3} \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} -\frac{5}{3} \\ 0 \\ 1 \end{bmatrix}$ are linearly independent, they form a basis.

(b) In fact we are dealing with the set

$$\left\{ \begin{bmatrix} 2t \\ -t \\ 4t \end{bmatrix} \mid t \in \mathbb{R} \right\} = \left\{ t \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \mid t \in \mathbb{R} \right\} = \text{span} \left\{ \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} \right\}.$$

Hence, the vector $\begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix}$ forms a basis.

10.16 Let $A$ be the matrix with as columns the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ and $\underline{v}_4$: $A = [\underline{v}_1 \quad \underline{v}_2 \quad \underline{v}_3 \quad \underline{v}_4]$.

Then $\text{span}\{\underline{v}_1, \ldots, \underline{v}_4\} = \text{Col}(A)$. Reduction of the matrix $A$ leads to

$$\begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 1 & 7 & 9 & 5 \\ 1 & 1 & 3 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 3 & 3 & 3 \\ 0 & 10 & 10 & 10 \\ 0 & 4 & 4 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & -1 & -5 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 & -2 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the columns 1 and 2 of this matrix contain the leading ones, the vectors $\underline{v}_1$ and $\underline{v}_2$ form a basis of the span of the given vectors.

Furthermore, $\underline{v}_3 = 2\,\underline{v}_1 + \underline{v}_2$ and $\underline{v}_4 = -2\,\underline{v}_1 + \underline{v}_2$.

10.18 (a) By using the fact that $T$ is a linear mapping, we obtain

$$T(\underline{e}_1) = T(\underline{v}_3) = \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix},$$

$$T(\underline{e}_2) = T(\underline{v}_2 - \underline{v}_3) = T(\underline{v}_2) - T(\underline{v}_3) = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} -1 \\ 5 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ -5 \\ 0 \end{bmatrix}$$

and

$$T(\underline{e}_3) = T(\underline{v}_1 - \underline{v}_2) = T(\underline{v}_1) - T(\underline{v}_2) = \begin{bmatrix} 2 \\ -1 \\ 4 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ -1 \\ 3 \end{bmatrix}.$$

(b) According to part (a), the standard matrix of the mapping $T$ is

$$A = [\,T(\underline{e}_1 \quad T(\underline{e}_2) \quad T(\underline{e}_3)\,] = \begin{bmatrix} -1 & 4 & -1 \\ 5 & -5 & -1 \\ 1 & 0 & 3 \end{bmatrix}.$$

(c) Since $\text{Ker}(T) = \text{Null}(A)$, we reduce the matrix $A$

$$\begin{bmatrix} -1 & 4 & -1 \\ 5 & -5 & -1 \\ 1 & 0 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 1 \\ 0 & 15 & -6 \\ 0 & 4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & -4 & 1 \\ 0 & 1 & \frac{1}{2} \\ 0 & 15 & -6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 \\ 0 & 1 & \frac{1}{2} \\ 0 & 0 & -13\frac{1}{2} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Hence, $\text{Ker}(T) = \{\underline{0}\}$.

(d) Since all the columns of the reduced matrix contain a leading one, all the columns of the matrix $A$ form a basis of $\text{Im}(T)$.

10.19 (a) Note that

$$MP = [\,M\underline{p}_1 \quad M\underline{p}_2 \quad M\underline{p}_3\,] = [\,\underline{p}_1 \quad 0.85\,\underline{p}_2 \quad 0.75\,\underline{p}_3\,]$$

and, if $\underline{d}_i$ denotes the $i$th column of $D$,

$$PD = [\,P\underline{d}_1 \quad P\underline{d}_2 \quad P\underline{d}_3\,] = [\,\underline{p}_1 \quad 0.85\,\underline{p}_2 \quad 0.75\,\underline{p}_3\,].$$

(b) For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: $M^n = PD^nP^{-1}$.

(1) According to part (a) and the invertibility of matrix $P$,

$$MP = PD \Longrightarrow (MP)P^{-1} = (PD)P^{-1} \Longrightarrow MPP^{-1} = PDP^{-1} \Longrightarrow M = PDP^{-1}.$$

Hence, the statement $\mathcal{P}(1)$ is true: $M = PDP^{-1}$.

(2) Let $k \in \mathbb{N}$ and assume that $\mathcal{P}(k)$ is true, that is: $M^k = PD^kP^{-1}$.

Then

$$M^{k+1} = M^kM = \left[PD^kP^{-1}\right]\left[PDP^{-1}\right] = PD^kP^{-1}PDP^{-1} = PD^kDP^{-1} = PD^{k+1}P^{-1}.$$

This proves that $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

(c) As $D$ is a diagonal matrix,

$$D^n = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.85^n & 0 \\ 0 & 0 & 0.75^n \end{bmatrix}.$$

As $M^n = PD^nP^{-1}$, we first determine the matrices $P$ and $P^{-1}$.

By definition, $P = \begin{bmatrix} \underline{p}_1 & \underline{p}_2 & \underline{p}_3 \end{bmatrix}$, where $\underline{p}_1$, $\underline{p}_2$ and $\underline{p}_3$ are the eigenvalues of the matrix $M$. According to the example,

$$\underline{p}_1 = \begin{bmatrix} 5 \\ 1.5 \\ 1 \end{bmatrix}.$$

As $\underline{p}_2$ is an eigenvector corresponding to the eigenvalue $\lambda = 0.85$, we find $\underline{p}_2$ by solving the system $(M - 0.85I)\underline{x} = \underline{0}$. Reduction leads to

$$\begin{bmatrix} 0.10 & 0.10 & 0.10 \\ 0.05 & -0.05 & 0.05 \\ 0 & 0.1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

So,

$$\underline{p}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

As $\underline{p}_3$ is an eigenvector corresponding to the eigenvalue $\lambda = 0.75$, we find $\underline{p}_3$ by solving the system $(M - 0.75I)\underline{x} = \underline{0}$. Reduction leads to

$$\begin{bmatrix} 0.20 & 0.10 & 0.10 \\ 0.05 & 0.05 & 0.05 \\ 0 & 0.1 & 0.1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 1 \\ 0 & 0.5 & 0.5 \\ 0 & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So,

$$\underline{p}_3 = \begin{bmatrix} 0 \\ -1 \\ 1 \end{bmatrix}.$$

In order to find the inverse of the matrix $P$ we reduce the matrix $\begin{bmatrix} P & I \end{bmatrix}$:

$$\begin{bmatrix} 5 & -1 & 0 & 1 & 0 & 0 \\ 1.5 & 0 & -1 & 0 & 1 & 0 \\ 1 & 1 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 3 & 0 & -2 & 0 & 2 & 0 \\ 5 & -1 & 0 & 1 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & -3 & -5 & 0 & 2 & -3 \\ 0 & -6 & -5 & 1 & 0 & -5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 1 & 1 & 0 & 0 & 1 \\ 0 & 1 & \frac{5}{3} & 0 & -\frac{2}{3} & 1 \\ 0 & 1 & \frac{5}{6} & -\frac{1}{6} & 0 & \frac{5}{6} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 & -\frac{2}{3} & 1 \\ 0 & 0 & -\frac{5}{6} & -\frac{1}{6} & \frac{2}{3} & -\frac{1}{6} \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -\frac{2}{3} & 0 & \frac{2}{3} & 0 \\ 0 & 1 & \frac{5}{3} & 0 & -\frac{2}{3} & 1 \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{4}{5} & \frac{1}{5} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{2}{15} & \frac{2}{15} & \frac{2}{15} \\ 0 & 1 & 0 & -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ 0 & 0 & 1 & \frac{1}{5} & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}.$$

As

$$P^{-1} = \begin{bmatrix} \frac{2}{15} & \frac{2}{15} & \frac{2}{15} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{5} \end{bmatrix},$$

we obtain

$$M^n = PD^nP^{-1} = \begin{bmatrix} 5 & -0.85^n & 0 \\ \frac{3}{2} & 0 & -0.75^2 \\ 1 & 0.85^n & 0.75^n \end{bmatrix} \begin{bmatrix} \frac{2}{15} & \frac{2}{15} & \frac{2}{15} \\ -\frac{1}{3} & \frac{2}{3} & \frac{2}{3} \\ \frac{1}{5} & -\frac{4}{5} & \frac{1}{5} \end{bmatrix}$$

$$= \begin{bmatrix} \frac{2}{3} + \frac{1}{3} \times 0.85^n & \frac{2}{3} - \frac{2}{3} \times 0.85^n & \frac{2}{3} - \frac{2}{3} \times 0.85^n \\ \frac{1}{5} - \frac{1}{5} \times 0.75^n & \frac{1}{5} + \frac{4}{5} \times 0.75^n & \frac{1}{5} - \frac{1}{5} \times 0.75^n \\ \frac{2}{15} - \frac{1}{3} \times 0.85^n + \frac{1}{5} \times 0.75^n & \frac{2}{15} + \frac{2}{3} \times 0.85^n - \frac{4}{5} \times 0.75^n & \frac{2}{15} + \frac{2}{3} \times 0.85^n + \frac{1}{5} \times 0.75^n \end{bmatrix}.$$

10.20 As $A = PDP^{-1}$ for some invertible matrix $P$ and some diagonal matrix $D$, it follows that $AP = PD$. If we indicate the columns of matrix $P$ by $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n$ and if we assume that $\lambda_1, \ldots, \lambda_n$ are the numbers on the main diagonal of $D$, then it holds that

$$AP = [\, A\underline{p}_1 \quad A\underline{p}_2 \quad \cdots \quad A\underline{p}_n \,]$$

and

$$PD = [\, \lambda_1\, \underline{p}_1 \quad \lambda_2\, \underline{p}_2 \quad \cdots \quad \lambda_n\, \underline{p}_n \,].$$

From this it follows that

$$A\underline{p}_1 = \lambda_1\, \underline{p}_1, \quad A\underline{p}_2 = \lambda_2\, \underline{p}_2, \ldots, A\underline{p}_n = \lambda_n\, \underline{p}_n.$$

Since $P$ is invertible, none of the columns of $P$ is a zero vector. This means that $\lambda_1, \lambda_2, \ldots, \lambda_n$ are eigenvalues of $A$, and that $\underline{p}_1, \underline{p}_2, \ldots, \underline{p}_n$ are the corresponding eigenvectors of $A$. Since $P$ is invertible, Theorem 3 implies that the columns of $P$ are linearly independent. Therefore matrix $A$ has $n$ linearly independent eigenvectors.