11.4 The polynomials $p_1\colon x \to x$, $p_2\colon x \to x^2$ and $p_3\colon x \to x^3$ form a basis of the subspace V. They are linearly independent and $V = \text{span}\{p_1, p_2, p_3\}$. So $\dim V = 3$.

11.5 Note that

$$W = \left\{ a \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \right\}$$

$$= \text{span}\left\{ \begin{bmatrix} 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 6 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 4 \\ -1 \\ 5 \end{bmatrix} \right\}.$$

In order to check the linear independency of the four vectors that span the subspace W, we reduce the matrix that can be constructed by means of these vectors:

$$A = \begin{bmatrix} 1 & -3 & 6 & 0 \\ 5 & 0 & 0 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 15 & -30 & 4 \\ 0 & 1 & -2 & -1 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 6 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, the vectors in column 1, 2 and 4 of the matrix A form a basis of W. So $\dim W = 3$.

11.6 If $W = V$, then, obviously, $\dim W = \dim V$.

Now suppose that $\dim W = \dim V = n$.

Assume that the vectors $\underline{w}_1, \ldots, \underline{w}_n$ form a basis of the subspace W. Then the vectors $\underline{w}_1, \ldots, \underline{w}_n$ are linearly independent vectors contained in V. So, according to Theorem 5, these vectors also form a basis of the space V. Hence, $W = \text{span}\{\underline{w}_1, \ldots, \underline{w}_n\} = V$.

11.7 We give a counterexample. Let

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \end{bmatrix}.$$

Then the matrix is fully reduced, the number of rows is $m = 2$ and the number of nonzero rows is $k = 2$. So $m - k = 0$. However, $\dim \text{Null}(A) = 1 \neq 0$.

On the contrary, according to Theorem 9, $\dim \text{Null(A)} = n - k$.

11.8 (a) Because $\mathbb{P} \subseteq \mathbb{F}$ is non-empty and the sum of two polynomial and a multiple of a polynomial again is a polynomial, $\mathbb{P}$ is a linear subspace of the space $\mathbb{F}$.

(b) Let, for $i \in \mathbb{N}$, p_i be the polynomial defined by

$$p_i(x) = x^i.$$

Further, p_0 is the polynomial defined by $p_0(x) = 1$.

These polynomials are in $\mathbb{P}$ are linearly independent as we will prove now.

If $c_0 p_0 + c_1 p_1 + \cdots = 0$, for some numbers $c_0, c_1, \ldots$, then for all $x \in \mathbb{R}$

$$c_0 p_0(x) + c_1 p_1(x) + \cdots = 0 \Longrightarrow c_0 + c_1 x + c_2 x^2 + \cdots = 0.$$

Hence, $c_0 = c_1 = \ldots = 0$.

Also these polynomials span the space $\mathbb{P}$.

Let $p \in \mathbb{P}$, say p is a polynomial of degree $n \in \mathbb{N}$. Then numbers $a_0, a_1, \ldots, a_n$ exists such that, for all $x \in \mathbb{R}$,

$$p(x) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n = a_0 p_0(x) + \cdots + a_n p_n(x).$$

So $p = a_0 p_0 + \cdots + a_n p_n$.

11.9 (Partial) reduction of the matrix A leads to

$$\begin{bmatrix} 1 & 4 & 5 & 2 \\ 2 & 1 & 3 & 0 \\ -1 & 3 & 2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & -7 & -7 & -4 \\ 0 & 7 & 7 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 5 & 2 \\ 0 & 7 & 7 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Obviously, rank $A = 2$ and $\dim \text{Null}(A) = 4 - 2 = 2$. Note that the number of free variables is equal to 2.

11.10 (a) If the matrix A is invertible, then $\text{rank}(A) = 4$ en $\dim \text{Null}(A) = 4 - 4 = 0$.

(b) The rank of a matrix is equal to the number of nonzero rows in the reduced matrix. So the rank of a 3×5 matrix is at most equal to 3. In that case $\dim \text{Null}(A) = 5 - 3 = 2$.

(c) The rank of a matrix is at most equal to the number of leading ones in the reduced matrix. So the rank of a 5×3 matrix is at most equal to 3. In that case $\dim \text{Null}(A) = 3 - 3 = 0$.

11.12 If $t = 1$, the matrix is

$$\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

So rank $A = 1$. If $t \neq 1$, reduction leads to

$$\begin{bmatrix} 1 & 1 & t \\ 1 & t & 1 \\ t & 1 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & t \\ 0 & t-1 & 1-t \\ 0 & 1-t & 1-t^2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & t \\ 0 & 1 & -1 \\ 0 & 1 & 1+t \end{bmatrix} \to \begin{bmatrix} 1 & 0 & t+1 \\ 0 & 1 & -1 \\ 0 & 0 & 2+t \end{bmatrix}.$$

If $t = -2$, the reduced matrix is

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

and rank $A = 2$.

If $t \neq -2$, further reduction leads to the matrix

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So rank $A = 3$.