11.1 The proof consists of two parts.

(1) Assume that the vectors $\underline{v}_1, \underline{v}_2, \ldots, \underline{v}_k$ are linearly independent. We will show that the vectors $T(\underline{v}_1), \ldots, T(\underline{v}_k)$ are linearly independent.

Suppose that $c_1\,T(\underline{v}_1) + \cdots + c_k\,T(\underline{v}_k) = \underline{0}$, for some numbers $c_1, \ldots, c_k$.

Because the mapping T is a linear mapping,

$$\underline{0} = c_1\,T(\underline{v}_1) + \cdots + c_k\,T(\underline{v}_k) = T(c_1\,\underline{v}_1 + \cdots + c_k\,\underline{v}_k).$$

Because the mapping T is one-to-one, only the zero vector (in V) is mapped onto the zero vector (in $\mathbb{R}^n$). So we may conclude that

$$c_1\,\underline{v}_1 + \cdots + c_k\,\underline{v}_k = \underline{0}.$$

The fact that the vectors $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent, implies that $c_1 = \ldots = c_k = 0$. In other words: the vectors $T(\underline{v}_1), \ldots, T(\underline{v}_k)$ are linearly independent.

(2) Assume that the vectors $T(\underline{v}_1), \ldots, T(\underline{v}_k)$ are linearly independent. We will show that the vectors $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent.

Suppose that $c_1\,\underline{v}_1 + \cdots + c_k\,\underline{v}_k = \underline{0}$, for some numbers $c_1, \ldots, c_k$.

Then $T(c_1\,\underline{v}_1 + \cdots + c_k\,\underline{v}_k) = \underline{0}$ and the fact that the mapping T is linear, imply that

$$c_1\,T(\underline{v}_1) + \cdots + c_k\,T(\underline{v}_k) = \underline{0}.$$

Since the vectors $T(\underline{v}_1), \ldots, T(\underline{v}_k)$ are linearly independent, it follows that $c_1 = \ldots = c_k = 0$. In other words: the vectors $\underline{v}_1, \ldots, \underline{v}_k$ are linearly independent.

11.2 (a) If $T\colon \mathbb{P}_2 \to \mathbb{R}^3$ is the coordinate mapping with respect to the usual basis of $\mathbb{P}_2$ formed by the polynomials $p_1\colon x \to 1$, $p_2\colon x \to x$ and $p_3\colon x \to x^2$, then

$$T(q_1) = T(p_1 + p_3) = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix},$$

$$T(q_2) = T(p_2 + p_3) = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$$

and

$$T(q_3) = T(p_1 + 2p_2 + p_3) = \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}.$$

(b) Note that the vectors $T(q_1), T(q_2)$ and $T(q_3)$ are linearly independent, because the matrix $A = [\,T(q_1) \quad T(q_2) \quad T(q_3)\,]$ is invertible, as can be proved as follows:

$$\det \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ 1 & 1 & 1 \end{bmatrix} = \det \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} + \det \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix} = -1 - 1 = -2 \neq 0.$$

Then, according to Exercise 11.1, the polynomials q_1, q_2 and q_3 are linearly independent.

(c) Because the matrix A is invertible, $\text{Col}(A) = \mathbb{R}^3$. Hence, the vectors $T(q_1), T(q_2)$ and $T(q_3)$, which correspond to the columns of the matrix A, span the space $\mathbb{R}^3$.

So if $p \in \mathbb{P}_2$, say $p\colon x \to a + bx + cx^2$, then $T(p)$ can be written as a linear combination of the vectors $T(q_1), T(q_2)$ and $T(q_3)$: say $T(p) = c_1\,T(q_1) + c_2\,T(q_2) + c_3\,T(q_3)$. This implies, by the linearity of the mapping T, that

$$T(p) = T\big(c_1\,q_1 + c_2\,q_2 + c_3\,q_3\big).$$

The fact that the mapping T is one-to-one implies that

$$p = c_1\,q_1 + c_2\,q_2 + c_3\,q_3.$$

This proves that the polynomials q_1, q_2 and q_3 span the space $\mathbb{P}_2$.

(d) Let q be the polynomial in $\mathbb{P}_2$ defined by $q(x) = 1 + 4x + 7x^2$.

In order to write the vector $T(q)$ as a linear combination of the vectors $T(q_1), T(q_2)$ and $T(q_3)$, we reduce the matrix

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 1 & 1 & 1 & 7 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 1 & 0 & 6 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & -2 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 4 \\ 0 & 0 & 1 & -1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & 6 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

So $T(q) = 2\,T(q_1) + 6\,T(q_2) - T(q_3) = T(2q_1 + 6q_2 - q_3)$. Since the mapping T is one-to-one, this implies that $q = 2q_1 + 6q_2 - q_3$. So the coordinate vector of q with respect to the basis of $\mathbb{P}_2$ formed by the polynomials q_1, q_2 and q_3 is

$$\begin{bmatrix} 2 \\ 6 \\ -1 \end{bmatrix}.$$

11.3 The proof consists of three parts.

(1) Let $\underline{u}$ and $\underline{u}'$ be vectors in U. Then, the fact that the mappings S and T are linear implies that

$$\begin{aligned}(T \circ S)(\underline{u} + \underline{u}') &= T\big(S(\underline{u} + \underline{u}')\big) = T\big(S(\underline{u}) + S(\underline{u}')\big) = T\big(S(\underline{u})\big) + T\big(S(\underline{u}')\big) \\ &= (T \circ S)(\underline{u}) + (T \circ S)(\underline{u}').\end{aligned}$$

Similarly, for all vectors $\underline{u}$ in U and all scalars c,

$$(T \circ S)(c\,\underline{u}) = c\,(T \circ S)(\underline{u}).$$

Hence, the mapping $T \circ S$ is linear.

(2) Assume that $(T \circ S)(\underline{u}) = (T \circ S)(\underline{u}')$.

Then $T\big(S(\underline{u})\big) = T\big(S(\underline{u}')\big)$. As the mapping T is one-to-one, this implies that $S(\underline{u}) = S(\underline{u}')$. As the mapping S is one-to-one, this implies that $\underline{u} = \underline{u}'$.

Hence, the mapping $T \circ S$ is one-to-one.

(3) Assume that $\underline{w}$ is a vector in W. We will show that this vector is the image under the mapping $T \circ S$ of at least one vector in the vector space U.

As the mapping T is surjective, there exists at least one vector, say $\underline{v}$, in the vector space V such that $T(\underline{v}) = \underline{w}$. As the mapping S is surjective, there exists at least one vector, say $\underline{u}$, in the vector space U such that $S(\underline{u}) = \underline{v}$. Then, however,

$$(T \circ S)(\underline{u}) = T\big(S(\underline{u})\big) = T(\underline{v}) = \underline{w}.$$

In other words: the mapping $T \circ S$ is surjective.

According to (1), (2) and (3), the mapping $T \circ S$ is an isomorphism.

11.11 Because $\dim \mathrm{Col}(A) \leq n$ and $\dim \mathrm{Row}(\mathrm{A}) \leq m$, the rank of a matrix is at most equal to $\min\{m, n\}$.
If $n \leq m$, then rank $A \leq n$ and $\dim \mathrm{Null}(A) \geq n - n = 0$.
If $m \leq n$, then rank $A \leq m$ and $\dim \mathrm{Null}(A) \geq n - m$.

11.13 (a) By using the definition Q we find that

$$Q^2 = (I - P)^2 = (I - P)(I - P) = I - P - PI + P^2 = I - 2P + P = I - P = Q$$

and

$$QP = (I - P)P = P - P^2 = P - P = O.$$

(b) If $\underline{y} \in \mathrm{Col}(P)$, then there exists an $\underline{x}$ such that $\underline{y} = P\underline{x}$. This however means that

$$Q\underline{y} = (I - P)\underline{y} = \underline{y} - P\underline{y} = P\underline{x} - P^2\underline{x} = P\underline{x} - P\underline{x} = \underline{0}.$$

So $\underline{y} \in \mathrm{Null}(Q)$.
If $\underline{y} \in \mathrm{Null}(Q)$, then

$$Q\underline{y} = \underline{0} \Longrightarrow (I - P)\underline{y} = \underline{0} \Longrightarrow \underline{y} - P\underline{y} = \underline{0} \Longrightarrow \underline{y} = P\underline{y}.$$

Hence $\underline{y} \in \mathrm{Col}(P)$.

(c) The Dimension Theorem and part (b) imply that

$$n = \mathrm{rank}(Q) + \dim\ \mathrm{Null}(Q) = \mathrm{rank}(Q) + \dim\ \mathrm{Col}(P) = \mathrm{rank}(Q) + \mathrm{rank}(P).$$