11.15 (a) According to Theorem 10, the system $A\underline{x} = \underline{b}$ is solvable.

In view of the Dimension Theorem, $\dim \text{Null}(A) = 3 - \text{rank } A = 0$.

The number of free variables in the system $A\underline{x} = \underline{0}$ equals $\dim \text{Null}(A) = 0$.

(b) According to Theorem 10, the system $A\underline{x} = \underline{b}$ is solvable.

In view of the Dimension Theorem, $\dim \text{Null}(A) = 9 - \text{rank } A = 7$.

The number of free variables in the system $A\underline{x} = \underline{0}$ equals $\dim \text{Null}(A) = 7$.

(c) According to Theorem 10, the system $A\underline{x} = \underline{b}$ is solvable.

In view of the Dimension Theorem, $\dim \text{Null}(A) = 3 - \text{rank } A = 2$.

The number of free variables in the system $A\underline{x} = \underline{0}$ equals $\dim \text{Null}(A) = 2$.

11.17 (a) We are considering a linear mapping $T: V \to W$, where $V = \mathbb{R}^5$. So $\dim V = 5$. According to the Dimension Theorem for linear mappings, $\dim \text{Ker}(T) = 5 - \dim \text{Im}(T) = 5 - 3 = 2$.

(b) We are considering a linear mapping $T: V \to W$, where $V = \mathbb{P}_4$. So $\dim V = 5$. According to the Dimension Theorem for linear mappings, $\dim \text{Ker}(T) = 5 - \dim \text{Im}(T) = 5 - 1 = 4$.

11.19 (a) If $\dim \text{Ker}(T) = 0$, then $\text{Ker}(T) = \{\underline{0}\}$. According to Theorem 5.3, in this case, the linear mapping T is one-to-one.

(b) Note that

$$\dim \text{Im}(T) = n - 1 < n = \dim V,$$

which implies that $\text{Im}(T) \neq V$. Then Exercise 18 implies that the mapping T is not one-to-one.

(c) Note that $\text{Im}(T) \subseteq \mathbb{R}^n$, which implies that $\dim \text{Im}(T) \leq n$.

In combination with Theorem 12 this leads to

$$\dim \text{Ker}(T) = \dim \mathbb{R}^m - \dim \text{Im}(T) = m - \dim \text{Im}(T) \geq m - n > 0.$$

So $\text{Ker}(T) \neq \{\underline{0}\}$. In view of Exercise 9.16, this means that the mapping T is not one-to-one.

(d) According to Exercise 18, the linear mapping is one-to-one.

11.20 Let $T: V \to W$ be a linear mapping. According to the Dimension Theorem for linear mappings,

$$\dim \text{Ker}(T) = \dim V - \dim \text{Im}(T) \geq \dim V - \dim W > 0.$$

Note that $\dim \text{Im}(T) \leq \dim W$, because $\text{Im}(T) \subset W$. So there exists at least one vector $\underline{v} \neq \underline{0}$ in $\text{Ker}(T)$. This however means that $T(\underline{v}) = \underline{0} = T(\underline{0})$. By consequence, the mapping T is not one-to-one.

12.1 Let $\underline{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ with $a, b, c \in \mathbb{R}$. Then

$$\underline{v}_1 \cdot \underline{v}_3 = 0 \Longleftrightarrow \frac{a}{\sqrt{6}} + \frac{b}{\sqrt{6}} - \frac{2c}{\sqrt{6}} = 0 \Longleftrightarrow a + b - 2c = 0,$$

$$\underline{v}_2 \cdot \underline{v}_3 = 0 \Longleftrightarrow \frac{a}{\sqrt{2}} - \frac{b}{\sqrt{2}} = 0 \Longleftrightarrow a - b = 0$$

and $$\|\underline{v}_3\| = 1 \Longleftrightarrow a^2 + b^2 + c^2 = 1.$$

The first two equations lead to $a = b = c$. In combination with the third equation this leads to $3a^2 = 1$. So $a = \frac{1}{\sqrt{3}}$. Hence, $\underline{v}_3 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

12.2 Note that

$$\underline{v}_1 \cdot \underline{v}_2 = \underline{v}_1 \cdot \underline{v}_3 = \underline{v}_2 \cdot \underline{v}_3 = 0$$

and

$$\underline{v}_1 \cdot \underline{v}_1 = \underline{v}_2 \cdot \underline{v}_2 = \underline{v}_3 \cdot \underline{v}_3 = 1.$$

So the vectors $\underline{v}_1, \underline{v}_2, \underline{v}_3$ are orthonormal.

Then, in view of Theorem 1, these vectors are also linearly independent.

So, in view of Theorem 11.5, these three vectors (in the space $\mathbb{R}^3$ of dimension 3) form a basis of $\mathbb{R}^3$.

The coordinates of the vector $\underline{u}$ with respect to this basis are

$$c_1 = \underline{u} \cdot \underline{v}_1 = 1,$$

$$c_2 = \underline{u} \cdot \underline{v}_2 = 0$$

and

$$c_3 = \underline{u}, \cdot \underline{v}_3 = \sqrt{2},$$

respectively. Hence, $\underline{u} = \underline{v}_1 + \sqrt{2}\, \underline{v}_3$.

12.6 Assume that $\underline{u} \in W^\perp$, that is: $\underline{u} \perp \underline{w}$ for all $\underline{w} \in W$. So, in particular, $\underline{u} \perp \underline{w}_i$ for all $1 \le i \le k$.

Now assume that $\underline{u} \perp \underline{w}_i$ for all $1 \le i \le k$.

Let $\underline{w} \in W$. Because the vectors $\underline{w}_1, \ldots, \underline{w}_k$ span the space W, $\underline{w} = c_1\, \underline{w}_1 + \cdots + c_k\, \underline{w}_k$, for some numbers $c_1, \ldots, c_k$. Hence,

$$\underline{u} \cdot \underline{w} = \underline{u} \cdot \big(c_1\, \underline{w}_1 + \cdots + c_k\, \underline{w}_k\big) = c_1\big(\underline{u} \cdot \underline{w}_1\big) + \cdots + c_k\big(\underline{u} \cdot \underline{w}_k\big) = 0.$$

This means that $\underline{u} \perp \underline{w}$. Because $\underline{w} \in W$ was arbitrarily chosen, $\underline{u} \perp \underline{w}$ for all $\underline{w} \in W$. So $\underline{u} \in W^\perp$.

12.7 We will prove that $\{\underline{0}\} \subset W \cap W^\perp$ and that $W \cap W^\perp \subset \{\underline{0}\}$.

Since W and $W^\perp$ are linear subspaces, $\underline{0} \in W$ and $\underline{0} \in W^\perp$. Hence, $\{\underline{0}\} \subset W \cap W^\perp$.

Assume that $\underline{u} \in W \cap W^\perp$. Then $\underline{u} \perp \underline{w}$ for all $\underline{w} \in W$. So $\underline{u} \perp \underline{u}$ and

$$\|\underline{u}\| = \sqrt{\underline{u} \cdot \underline{u}} = 0.$$

Hence, $\underline{u} = \underline{0}$. By consequence, $W \cap W^\perp \subset \{\underline{0}\}$.

12.8 Note that

$$\begin{aligned} W^\perp &= \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| \begin{bmatrix} x \\ y \\ z \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -5 \\ 4 \end{bmatrix} = 0 \right\} = \left\{ \begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 \middle| \; 2x - 5y + 4z = 0 \right\} \\ &= \left\{ \begin{bmatrix} \frac{5}{2}y - 2z \\ y \\ z \end{bmatrix} \middle| \; y, z \in \mathbb{R} \right\} = \left\{ y \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \middle| \; y, z \in \mathbb{R} \right\} = \operatorname{span} \left\{ \begin{bmatrix} \frac{5}{2} \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} \right\}. \end{aligned}$$

12.9 In fact we are looking for a basis of the orthogonal complement of the linear space $W = \text{Col}(A)$, where

$$A = \begin{bmatrix} 2 & 4 \\ 0 & 0 \\ -1 & -2 \end{bmatrix}.$$

According to Theorem 4, $W^\perp = \text{Null}(A^T)$. Partial reduction of the matrix A^T leads to

$$\begin{bmatrix} 2 & 0 & -1 \\ 4 & 0 & -2 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So every vector in $\text{Null}(A^T)$ can be written as

$$\begin{bmatrix} \frac{1}{2}z \\ y \\ z \end{bmatrix} = z \begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

where $y, z \in \mathbb{R}$. Hence, the vectors $\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}$ span the space $\text{Null}(A^T) = W^\perp$. Because these vectors are linearly independent too (please check), they form a basis of the space in question.