11.14 We will show that the three statements are equivalent by proving that $(a) \Rightarrow (b), (b) \Rightarrow (c)$ and $(c) \Rightarrow (a)$.

(a) $\Rightarrow$ (b)

Assume that the system $A\underline{x} = \underline{b}$ is solvable for every vector $\underline{b} \in \mathbb{R}^m$. Then each vector $\underline{b} \in \mathbb{R}^m$ is a linear combination of the columns of the matrix A. Hence, the columns of the matrix A span the space $\mathbb{R}^m$.

(b) $\Rightarrow$ (c)

Assume that the columns of the matrix A span the space $\mathbb{R}^m$. Because $\text{Col}(A) = \text{span}\{\underline{a}_1, \ldots, \underline{a}_n\} = \mathbb{R}^m$,

$$\text{rank } A = \dim \text{Col}(A) = \dim \mathbb{R}^m = m.$$

(c) $\Rightarrow$ (a)

Assume that rank $A = m$. In order to show that the system $A\underline{x} = \underline{b}$ is solvable for every $\underline{b} \in \mathbb{R}^m$, we choose a vector $\underline{b} \in \mathbb{R}^m$.

Because $\dim \text{Col}(A) = \text{rank } A = m$, $\text{Col}(A)$ is a linear subspace of the space $\mathbb{R}^m$ of dimension m. Hence, in view of Exercise 6, $\text{Col}(A) = \mathbb{R}^m$. In other words: $\mathbb{R}^m = \text{span}\{\underline{a}_1, \ldots, \underline{a}_n\}$.

Then, however, $\underline{b}$ can be written as a linear combination of the columns of the matrix A. So the system $A\underline{x} = \underline{b}$ is solvable.

11.18 Assume that the mapping T is one-to-one.

In view of Exercise 9.16, the mapping T is one-to-one if and only if $\text{Ker}(T) = \{\underline{0}\}$. According to the Dimension Theorem for Linear Mappings, this is the case if and only if $\dim \text{Im}(T) = \dim V$.

In other words: the mapping T is one-to-one if and only if $\dim \text{Im}(T) = \dim V$.

Because $\text{Im}(T)$ is a linear subspace of the space V, $\dim \text{Im}(T) = \dim V$ if and only if $\text{Im}(T) = V$.

12.3 We introduce, for $m \in \mathbb{N}$, the statement $\mathcal{P}(m)$: if the vectors $\underline{v}_1, \ldots, \underline{v}_m$ in $\mathbb{R}^n$ are pairwise orthogonal, then

$$\| \underline{v}_1 + \cdots + \underline{v}_m \|^2 = \| \underline{v}_1 \|^2 + \cdots + \| \underline{v}_m \|^2.$$

(1) Obviously, the statement $\mathcal{P}(1)$ is true.

(2) Let $k \in \mathbb{N}$ and assume that the statement $\mathcal{P}(k)$ is true, that is:

$\| \underline{v}_1 + \cdots + \underline{v}_k \|^2 = \| \underline{v}_1 \|^2 + \cdots + \| \underline{v}_k \|^2$.

If the vectors $\underline{v}_1, \ldots, \underline{v}_{k+1}$ are pairwise orthogonal, then the fact that

$$(\underline{v}_1 + \cdots + \underline{v}_k) \cdot \underline{v}_{k+1} = \underline{v}_1 \cdot \underline{v}_{k+1} + \cdots + \underline{v}_k \cdot \underline{v}_{k+1} = 0$$

implies that $\underline{v}_1 + \cdots + \underline{v}_k \perp \underline{v}_{k+1}$.

So, according to the Theorem of Pythagoras,

$$\begin{aligned} \| (\underline{v}_1 + \cdots + \underline{v}_k) + \underline{v}_{k+1} \|^2 &= \| \underline{v}_1 + \cdots + \underline{v}_k \|^2 + \| \underline{v}_{k+1} \|^2 \\ &= \| \underline{v}_1 \|^2 + \cdots + \| \underline{v}_k \|^2 + \| \underline{v}_{k+1} \|^2. \end{aligned}$$

Hence, the statement $\mathcal{P}(k+1)$ is true.

According to the Principle of Induction, the statement $P(m)$ is true for all $m \in \mathbb{N}$.

12.4 Note that the ith coordinate of the vector $\underline{u}$ with respect to the basis formed by the vectors $\underline{w}_1, \ldots, \underline{w}_m$ is given by $\underline{u} \cdot \underline{w}_i$. So $\underline{u} = \sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)\, \underline{w}_i$. Hence, according to Exercise 3,

$$\|\underline{u}\|^2 = \|\sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)\, \underline{w}_i\|^2 = \sum_{i=1}^{m} \|(\underline{u} \cdot \underline{w}_i)\, \underline{v}_i\|^2 = \sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)^2 \|\underline{v}_i\|^2 = \sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)^2.$$

Alternative:

$$\|\underline{u}\|^2 = \underline{u} \cdot \underline{u} = \underline{u} \cdot \left[\sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)\, \underline{w}_i\right] = \sum_{i=1}^{m} (\underline{u} \cdot \underline{w}_i)^2.$$

12.5 The entry at position (i, j) of the matrix $A^T A$ is obtained by combining the ith row of A^T and the jth column of A. To be precise: this entry is the dot product of the ith row of A^T and the jth column of A. Because the ith row of A^T is precisely the ith column of A, we find at position (i, j) of the matrix $A^T A$ the dot product of the ith column and the jth column of A.

Hence, the columns of A are orthonormal if and only if there is

- a 0 at position (i, j) of $A^T A$ if $i \neq j$ and
- a 1 if $i = j$.

That is: if and only if $A^T A = I$.

12.10 (a) Since $\text{Ker}(T) = \text{Null}(A)$, we reduce the matrix A

$$\begin{bmatrix} 1 & 0 & 2 \\ 0 & -1 & 1 \\ 2 & 1 & 3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

So $\underline{x} \in \text{Null}(A)$ if and only if $\underline{x}$ is a solution of the system $\begin{cases} x_1 + 2x_3 = 0 \\ x_2 - x_3 = 0 \end{cases}$, which is equivalent to

$$\underline{x} = \begin{bmatrix} -2x_3 \\ x_3 \\ x_3 \end{bmatrix} = x_3 \underbrace{\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}}_{=\underline{w}} \Longleftrightarrow \underline{x} \in \text{span}\{\underline{w}\}.$$

(b) Since $\text{Ker}(T) = \text{span}\{\underline{w}\}$,

$$\text{Ker}(T)^\perp = \{\underline{x} \mid \underline{x} \perp \text{span}\{\underline{w}\}\} = \{\underline{x} \mid \underline{x} \perp \underline{w}\} = \{\underline{x} \mid \langle \underline{x}, \underline{w} \rangle = 0\} = \{\underline{x} \mid -2x_1 + x_2 + x_3 = 0\}.$$

So any $\underline{x} \in \text{Ker}(T)^\perp$ has the form

$$\underline{x} = \begin{bmatrix} \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2 \\ x_3 \end{bmatrix} = x_2 \underbrace{\begin{bmatrix} \frac{1}{2} \\ 1 \\ 0 \end{bmatrix}}_{=\underline{u}} + x_3 \underbrace{\begin{bmatrix} \frac{1}{2} \\ 0 \\ 1 \end{bmatrix}}_{=\underline{v}}.$$

This proves that the vectors $\underline{u}$ and $\underline{v}$ span $\text{Ker}(T)^\perp$. Since these vectors are also linearly independent, they form a basis of $\text{Ker}(T)^\perp$.

(c) Note that $\mathrm{Im}(T) = \mathrm{Col}(A)$. According to part (a), the first two columns $\underline{a}_1$ and $\underline{a}_2$ of the matrix A form a basis of $\mathrm{Col}(A)$.

Obviously, $\underline{a}_1 = 2\,\underline{v}$ and $\underline{a}_2 = \underline{v} - \underline{u}$. Because the linear subspace $\mathrm{Ker}(T)^\perp$ is closed with respect to addition and scalar multiplication, this implies that $\underline{a}_1, \underline{a}_2 \in \mathrm{Ker}(T)^\perp$. Then, for the same reason, $\mathrm{Col}(A) \subset \mathrm{Ker}(T)^\perp$.

In a similar way, one proves the reverse inclusion. Hence, $\mathrm{Col}(A) = \mathrm{Ker}(T)^\perp$.