12.11 First we will prove that $W \subset \left(W^\perp\right)^\perp$.

Let $\underline{w} \in W$. Then $\underline{w} \perp \underline{v}$ for all $\underline{v} \in W^\perp$. So $\underline{w} \in \left(W^\perp\right)^\perp$. Since $\underline{w}$ was arbitrarily chosen, this means that $W \subset \left(W^\perp\right)^\perp$.

Next we will show that $\left(W^\perp\right)^\perp \subset W$.

Let $\underline{u} \in \left(W^\perp\right)^\perp$. Then, according to the Projection Theorem,

$$\underline{u} = \underline{u}' + \underline{u}'',$$

where $\underline{u}' \in W$ and $u'' \in W^\perp$. By consequence

$$0 = \underline{u} \cdot \underline{u}'' = \underline{u}' \cdot \underline{u}'' + \underline{u}'' \cdot \underline{u}'' = \underline{u}'' \cdot \underline{u}''.$$

Hence $\underline{u}'' = 0$, which implies that

$$\underline{u} = \underline{u} + \underline{u}'' = \underline{u}' \in W.$$

Since $\underline{u}$ was arbitrarily chosen, this means that $\left(W^\perp\right)^\perp \subset W$.

12.12 The orthogonal projection of the vector

$$\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

on W is

$$\text{proj}_W(\underline{u}) = \frac{\underline{u} \cdot \underline{w}_1}{\|\underline{w}_1\|^2}\,\underline{w}_1 + \frac{\underline{u} \cdot \underline{w}_2}{\|\underline{w}_2\|^2}\,\underline{w}_2 = \tfrac{9}{30}\,\underline{w}_1 + \tfrac{3}{6}\underline{w}_2 = \tfrac{3}{10}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} + \tfrac{1}{2}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix}.$$

The component of $\underline{u}$ which is orthogonal to W is given by

$$\underline{u} - \text{proj}_W(\underline{u}) = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} - \tfrac{1}{5}\begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix} = \tfrac{1}{5}\begin{bmatrix} 7 \\ 0 \\ 14 \end{bmatrix}.$$

12.14 Observe that the vectors $\underline{w}_1$ and $\underline{w}_2$ are orthogonal. Since

$$\text{proj}_W(\underline{e}_1) = \frac{\underline{e}_1 \cdot \underline{w}_1}{\|\underline{w}_1\|^2}\underline{w}_1 + \frac{\underline{e}_1 \cdot \underline{w}_2}{\|\underline{w}_2\|^2}\underline{w}_2 = \tfrac{2}{30}\underline{w}_1 + \tfrac{-2}{6}\underline{w}_2 = \tfrac{1}{15}\underline{w}_1 - \tfrac{5}{15}\underline{w}_2 = \tfrac{1}{15}\begin{bmatrix} 12 \\ 0 \\ -6 \end{bmatrix},$$

$$\text{proj}_W(\underline{e}_2) = \frac{\underline{e}_2 \cdot \underline{w}_1}{\|\underline{w}_1\|^2}\underline{w}_1 + \frac{\underline{e}_2 \cdot \underline{w}_2}{\|\underline{w}_2\|^2}\underline{w}_2 = \tfrac{5}{30}\underline{w}_1 + \tfrac{1}{6}\underline{w}_2 = \tfrac{1}{6}\underline{w}_1 + \tfrac{1}{6}\underline{w}_2 = \tfrac{1}{6}\begin{bmatrix} 0 \\ 6 \\ 0 \end{bmatrix}$$

and

$$\text{proj}_W(\underline{e}_3) = \frac{\underline{e}_3 \cdot \underline{w}_1}{\|\underline{w}_1\|^2}\underline{w}_1 + \frac{\underline{e}_3 \cdot \underline{w}_2}{\|\underline{w}_2\|^2}\underline{w}_2 = \tfrac{-1}{30}\underline{w}_1 + \tfrac{1}{6}\underline{w}_2 = -\tfrac{1}{30}\underline{w}_1 + \tfrac{5}{30}\underline{w}_2 = \tfrac{1}{30}\begin{bmatrix} -12 \\ 0 \\ 6 \end{bmatrix},$$

the standard matrix of the orthogonal projection proj_W is given by

$$A = \tfrac{1}{30}\begin{bmatrix} 24 & 0 & -12 \\ 0 & 30 & 0 \\ -12 & 0 & 6 \end{bmatrix}.$$

12.15 (a) If $A = [\underline{w}_1 \quad \underline{w}_2 \quad \cdots \quad \underline{w}_m]$, then

$$A^T \underline{u} = \begin{bmatrix} \underline{w}_1^T \\ \vdots \\ \underline{w}_m^T \end{bmatrix} \underline{u} = \begin{bmatrix} \underline{w}_1^T \underline{u} \\ \vdots \\ \underline{w}_m^T \underline{u} \end{bmatrix} = \begin{bmatrix} \underline{w}_1 \cdot \underline{u} \\ \vdots \\ \underline{w}_m \cdot \underline{u} \end{bmatrix}.$$

So

$$AA^T \underline{u} = \left(\underline{w}_1 \cdot \underline{u}\right) \underline{w}_1 + \cdots + \left(\underline{w}_m \cdot \underline{u}\right) \underline{w}_m = \mathrm{proj}_W(\underline{u}).$$

(b) Note that

$$\left(AA^T\right)^T = (A^T)^T A^T = AA^T.$$

(c) Note that the entry at the position (i, j) of the matrix $A^T A$ is

$$\underline{w}_i \cdot \underline{w}_j = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j. \end{cases}$$

Hence $A^T A = I$, so that

$$\left(AA^T\right)^2 = AA^T \cdot AA^T = A \cdot A^T A \cdot A^T = A \cdot I \cdot A^T = AA^T.$$