1.1 (a) According to the definitions,

$$\underline{u} + (-1)\underline{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (-1)\begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$$

and
$$5(2\,\underline{u} + \underline{w}) = 10\,\underline{u} + 5\,\underline{w} = 10\begin{bmatrix} 1 \\ 1 \end{bmatrix} + 5\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 10 \\ 10 \end{bmatrix} + \begin{bmatrix} -20 \\ 10 \end{bmatrix} = \begin{bmatrix} -10 \\ 20 \end{bmatrix}.$$

The lengths of these vectors are

$$\|\,\underline{u} + (-1)\underline{v}\,\| = \sqrt{(-1)^2 + 2^2} = \sqrt{5}$$

and
$$\|\,5(2\,\underline{u} + \underline{w})\,\| = \sqrt{(-10)^2 + 20^2} = 10\sqrt{5}.$$

(b) As

$$c\,\underline{u} + d\,\underline{v} = \begin{bmatrix} 5 \\ -1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} c + 2d \\ c - d \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \end{bmatrix},$$

it holds that $c + 2d = 5$ and $c - d = -1$. By subtracting these equations, we obtain $3d = 6$. So $d = 2$ and $c = 1$.

(c) As

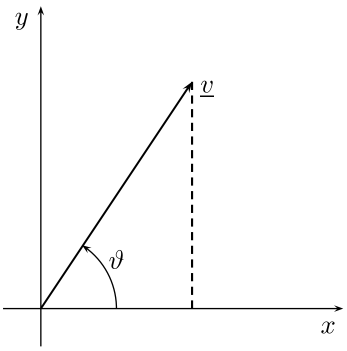
$$c\,\underline{v} + d\,\underline{w} = \underline{u} \Longleftrightarrow c\begin{bmatrix} 2 \\ -1 \end{bmatrix} + d\begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \Longleftrightarrow \begin{bmatrix} 2c - 4d \\ -c + 2d \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

it holds that $2c - 4d = 1$ and $-c + 2d = 1$. Since the first equation implies that $c - 2d = \frac{1}{2}$, whereas the second one implies that $c - 2d = -1$, there exist no numbers c and d satisfying the relation given in the exercise.

(d) Note that

$$\|\,c\,\underline{u}\,\| = 1 \Longleftrightarrow \sqrt{c^2 + c^2} = 1 \Longleftrightarrow 2c^2 = 1 \Longleftrightarrow c = \pm\tfrac{1}{2}\sqrt{2}.$$

1.2 Consider the triangle in the following figure



By using the definition of sinus and cosines, we find that

$$\sin\vartheta = \frac{v_2}{\|\,\underline{v}\,\|} \qquad \text{and} \qquad \cos\vartheta = \frac{v_1}{\|\,\underline{v}\,\|}.$$

Hence, $v_1 = \|\,\underline{v}\,\|\cos\vartheta$, whereas $v_2 = \|\,\underline{v}\,\|\sin\vartheta$.

1.3 According to the properties of the length,

$$\left\|\frac{1}{\|\,\underline{v}\,\|}\,\underline{v}\right\| = \left|\frac{1}{\|\,\underline{v}\,\|}\right|\|\,\underline{v}\,\| = \frac{1}{\|\,\underline{v}\,\|}\|\,\underline{v}\,\| = 1.$$

1.4 As

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\|\cos\vartheta \iff 1 = \|\underline{u}\|\|\underline{v}\|\cos\vartheta,$$

$\cos\vartheta > 0$. Hence, the vectors $\underline{u}$ and $\underline{v}$ make an acute angle.

As

$$\underline{v}\cdot\underline{w} = \|\underline{v}\|\|\underline{w}\|\cos\vartheta \iff 0 = \|\underline{v}\|\|\underline{w}\|\cos\vartheta,$$

$\cos\vartheta = 0$. Hence, the vectors $\underline{v}$ and $\underline{w}$ are orthogonal.

As

$$\underline{u}\cdot\underline{w} = \|\underline{u}\|\|\underline{w}\|\cos\vartheta \iff -3 = \|\underline{u}\|\|\underline{w}\|\cos\vartheta,$$

$\cos\vartheta < 0$. Hence, the vectors $\underline{u}$ and $\underline{w}$ make an obtuse angle.

1.5 (a) The two vectors $\underline{u}$ and $\underline{v}$ are parallel if $t = 2\times\frac{5}{3} = \frac{10}{3}$.

(b) The vectors $\underline{u}$ and $\underline{v}$ are orthogonal if

$$\underline{u}\cdot\underline{v} = 0 \iff 6+5t = 0 \iff t = -\tfrac{6}{5}.$$

(c) If the angle between the two vectors $\underline{u}$ and $\underline{v}$ is $\pi/4$, then

$$\underline{u}\cdot\underline{v} = \|\underline{u}\|\|\underline{v}\|\cos\pi/4 \iff 6+5t = \sqrt{4+t^2}\cdot\sqrt{9+25}\cdot\tfrac{1}{2}\sqrt{2} \iff (6+5t)^2 = (4+t^2)\cdot 34\cdot\tfrac{1}{2}$$

$$\iff 36+60t+25t^2 = 68+17t^2 \iff 2t^2+15t-8 = 0$$

$$\iff t = \frac{-15\pm\sqrt{225+64}}{4} = \frac{-15\pm 17}{4}.$$

So $t = \frac{1}{2}$ or $t = -8$.

1.7 (a) As

$$\underline{u}-\underline{v} = \begin{bmatrix}1\\1\end{bmatrix} - \begin{bmatrix}2\\3\end{bmatrix} = \begin{bmatrix}-1\\-2\end{bmatrix}$$

is a directional vector of the line and $\underline{u}$ is a support vector, a vector representation of the line is

$$\underline{x} = \begin{bmatrix}1\\1\end{bmatrix} + t\begin{bmatrix}-1\\-2\end{bmatrix}.$$

(b) As $\underline{p}-\underline{q}$ is a directional vector of the line and $\underline{p}$ is a support vector,

$$\underline{x} = \underline{p} + t(\underline{p}-\underline{q})$$

is a vector representation of the line through the (points corresponding to the) vectors $\underline{p}$ and $\underline{q}$.

1.8 A point on the given line is of the form $(2+t,t)$, for some number t. For example the points $(2,0)$ and $(3,1)$ are on the line (here we choose two points because a straight line is uniquely determined by choosing two different points on this line). If these points are on the line with algebraic equation $ax+by+c=0$, then $2a+c=0$ and $3a+b+c=0$. So $a = -\frac{1}{2}c$ and $b = -3a-c = \frac{1}{2}c$. So an algebraic equation of the line is (choose $c=-2$)

$$x-y-2=0.$$

1.9 As $x - y - 2 = 0 \Longleftrightarrow x - y = 2$, a normal equation of the line introduced in the foregoing exercise is $\underline{v} \cdot \underline{n} = 2$, where $\underline{n} = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

1.11 A vector representation of a line in $\mathbb{R}^3$ is

$$\underline{v} = \underline{s} + t\underline{r},$$

where $\underline{s}$ is a three-dimensional vector on the line and $\underline{r}$ is a three-dimensional vector determining the direction of the line.

1.13 (a) As the plane $\mathcal{Q}$ passes through the origin, $ax + by + cz = 0$ is an algebraic equation of this plane. As the points $(1, 2, 0)$ and $(-1, 0, 2)$ are on this plane, $a + 2b = 0$ and $-a + 2c = 0$. So $b = -\frac{1}{2}a$ and $c = \frac{1}{2}a$. By choosing $a = 2$ we obtain the equation

$$2x - y + z = 0.$$

(b) According to part (a), a normal equation of the plane $\mathcal{Q}$ is $\underline{v} \cdot \underline{n} = 0$, where $\underline{n} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}$.

Let $\mathcal{Q}$ be the (unique) plane through the origin and the points $(1, 2, 0)$ and $(-1, 0, 2)$.

(c) As

$$\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} - \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

is a directional vector of the line and $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ is a support vector,

$$\underline{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 2 \\ -2 \end{bmatrix}$$

is a vector representation of the line through the points $(1, 2, 0)$ and $(-1, 0, 2)$.

1.15 (a) Formally, the plane is the set

$$\begin{aligned} \mathcal{P} &= \{(x, y, z) \in \mathbb{R}^3 |\ x + 2y + 3z = 4\} = \{(4 - 2y - 3z, y, z) \in \mathbb{R}^3 |\ y, z \in \mathbb{R}\} \\ &= \left\{ \begin{bmatrix} 4 - 2y - 3z \\ y \\ z \end{bmatrix} \in \mathbb{R}^3 |\ y, z \in \mathbb{R} \right\} \\ &= \left\{ \begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} -2y \\ y \\ 0 \end{bmatrix} + \begin{bmatrix} -3z \\ 0 \\ z \end{bmatrix} \in \mathbb{R}^3 |\ y, z \in \mathbb{R} \right\} \\ &= \Bigg\{ \underbrace{\begin{bmatrix} 4 \\ 0 \\ 0 \end{bmatrix}}_{=\underline{s}} + y \underbrace{\begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}}_{=\underline{r}_1} + z \underbrace{\begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}}_{=\underline{r}_2} \in \mathbb{R}^3 |\ y, z \in \mathbb{R} \Bigg\}. \end{aligned}$$

The vector representation is

$$\underline{v} = \underline{s} + t_1\,\underline{r}_1 + t_2\,\underline{r}_2.$$

(b) A normal representation of the plane is

$$\underline{v} \cdot \underline{n} = 4,$$

where $\underline{n} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$.