1.6 (a) The line ℓ passes through the origin because the equation $ax + by = 0$ is satisfied when we substitute 0 for x and y.

(b) Let $\underline{v}$ be a vector on the line ℓ. Then

$$av_1 + bv_2 = 0 \Longleftrightarrow \begin{bmatrix} a \\ b \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = 0 \Longleftrightarrow \underline{v} \cdot \underline{n} = 0 \Longleftrightarrow \underline{v} \perp \underline{n}.$$

1.10 (a) If the line is vertical, then an equation is $x = 0$.

Otherwise, an equation is $y = mx$ with $m \in \mathbb{R}$.

Alternative: $ax + by = 0$, where $a \neq 0$ or $b \neq 0$.

(b) A vector representation of this line is $\underline{v} = t\,\underline{r}$ with $\underline{r}$ a nonzero vector.

(c) A normal representation is $\underline{v} \cdot \underline{n} = 0$, where $\underline{n}$ is a nonzero vector.

1.12 Let

$$\underline{v} = t_1\,\underline{r}_1 + t_2\,\underline{r}_2,$$

be a vector representation of the plane $\mathcal{P}$.

(a) Assume that $\underline{w}$ is orthogonal to all vectors in this plane. As $\underline{r}_1$ and $\underline{r}_2$ are vectors contained in the plane, $\underline{w}$ is orthogonal to these directional vectors.

(b) If the vector $\underline{w}$ is orthogonal to the directional vector $\underline{r}_1$ and $\underline{r}_2$ and $\underline{v}$ is an arbitrary vector in the plane, then

$$\underline{v} = t_1\,\underline{r}_1 + t_2\,\underline{r}_2,$$

for some numbers t_1 and t_2, so that

$$\underline{v} \cdot \underline{w} = \big(t_1\,\underline{r}_1 + t_2\,\underline{r}_2\big) \cdot \underline{w} = t_1\big(\underline{r}_1 \cdot \underline{w}\big) + t_2\big(\underline{r}_2 \cdot \underline{w}\big) = 0.$$

In other words: $\underline{w}$ is orthogonal to $\underline{v}$.

1.14 (a) A normal equation of the plane $\mathcal{P}$ is $\underline{v} \cdot \underline{n} = c$, for some number c.

As the plane contains the point $(-1, 2, 2)$,

$$-1 \cdot 1 + 2 \cdot 1 + 2 \cdot -1 = c \Longleftrightarrow c = -1.$$

So $\underline{v} \cdot \underline{n} = -1$ is a normal equation of the plane.

(b) If we write $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for a vector in the plane $\mathcal{P}$, then according to part (a),

$$\underline{v} \cdot \underline{n} = -1 \Longleftrightarrow x + y - z = -1.$$

So $x + y - z = -1$ is an algebraic equation of the plane $\mathcal{P}$.

(c) According to part (b), a vector $\underline{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is in the plane $\mathcal{P}$ if and only if $z = x + y + 1$. So a vector is in the plane $\mathcal{P}$ if and only if it is of the form

$$\begin{bmatrix} x \\ y \\ x+y+1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} x \\ 0 \\ x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + x \underbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}_{=\underline{r}_1} + y \underbrace{\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}}_{=\underline{r}_2},$$

for some numbers x and y. So $\underline{r}_1$ and $\underline{r}_2$ are directional vectors of the plane $\mathcal{P}$ and $\underline{n} \cdot \underline{r}_1 = \underline{n} \cdot \underline{r}_2 = 0$, that is: $\underline{n}$ is orthogonal to these directional vectors.

1.16 (a) In order to find a normal representation of the two planes we determine for each plane a normal vector.

If $\underline{n}_1 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a normal vector of the plane $\mathcal{P}_1$, then $\underline{n}_1$ is orthogonal to the two directional vectors of this plane:

$$\begin{cases} y - z = 0 \\ x - z = 0 \end{cases} \Longrightarrow x = y = z \Longrightarrow \underline{n}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}.$$

As $\underline{n}_1 \cdot \underline{s}_1 = 2$, a (normal and algebraic) representation of the first plane is

$$\underline{v} \cdot \underline{n}_1 = 2 \Longleftrightarrow x + y + z = 2.$$

If $\underline{n}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is a normal vector of the plane $\mathcal{P}_2$, then $\underline{n}_2$ is orthogonal to the two directional vectors of this plane:

$$\begin{cases} y + z = 0 \\ -x + z = 0 \end{cases} \Longrightarrow x = z \quad \text{and} \quad y = -z \Longrightarrow \underline{n}_2 = \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

As $\underline{n}_2 \cdot \underline{s}_2 = 1$, a (normal and algebraic) representation of the second plane is

$$\underline{v} \cdot \underline{n}_2 = 1 \Longleftrightarrow x - y + z = 1.$$

A point $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is on the intersection of the two planes if

$$\begin{cases} x + y + z = 2 \\ x - y + z = 1. \end{cases}$$

If we subtract the two equations we get $y = \frac{1}{2}$. Then according to the first equation, $x = 2 - y - z = 1\frac{1}{2} - z$

So each point on the intersection can be written as

$$\begin{bmatrix} 1\frac{1}{2} - z \\ \frac{1}{2} \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1\frac{1}{2} \\ \frac{1}{2} \\ 0 \end{bmatrix}}_{=\underline{s}} + z \underbrace{\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}}_{=\underline{r}},$$

for some $z \in \mathbb{R}$.

Hence, the intersection of the two planes is a line with vector representation $\underline{v} = \underline{s} + t\,\underline{r}$.

(b) The two planes can be represented by the equations $n_1x + n_2y + n_3z = a$ and $n_1x + n_2y + n_3z = b$, where n_1, n_2 and n_3 are the coordinates of the vector $\underline{n}$ and where a and b are numbers.

If $a = b$, then the two planes coincide (because they have the same algebraic representation).

If $a \neq b$, then the system

$$\begin{cases} n_1x + n_2y + n_3z = a \\ n_1x + n_2y + n_3z = b \end{cases}$$

has no solution. That is: the two planes have nothing in common.