2.1 The equations in $(a), (c)$ and (f) are linear.

2.2 Substitution of the given numbers for the variables in the third equation of the system leads to $5 = -7$. So the third equation is not satisfied. Hence the given sequence of numbers is not a solution of the system.

2.3 The first equation leads to $x = y + 3$. If we replace in the second equation x by $y + 3$, then we find that

$$2(y + 3) - 2y = k \Longleftrightarrow k = 6.$$

So the system has no solutions if $k \neq 6$.

There is no k such that the system has a unique solution.

The system has an infinite number of solutions if $k = 6$. Each solution is of the form

$$\begin{cases} x = 3 + t \\ y = t, \end{cases}$$

where $t \in \mathbb{R}$.

2.4 An example of a system is

$$\begin{cases} 7x + 2y + z - 3u = 5 \\ x + 2y + 4z = 1 \end{cases}$$

2.5 The standard form of the system is

$$\begin{cases} 2x_1 + 2x_3 = 1 \\ 3x_1 + x_2 + 4x_3 = -7 \\ 6x_1 + x_2 - x_3 = 0. \end{cases}$$

Hence the augmented coefficient matrix is

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 3 & 1 & 4 & -7 \\ 6 & 1 & -1 & 0 \end{bmatrix}.$$

2.6 (a) The matrix has the reduced row-echelon form.

(b) Because property (3) is not satisfied, the matrix has no (reduced) row-echelon form.

(c) The matrix has the reduced row-echelon form.

(d) The matrix has the row-echelon form.

(e) Because property (4) is not satisfied, the matrix has no (reduced) row-echelon form.

(f) The matrix has the reduced row-echelon form.

2.7 (a) The leading variables are x_1, x_2 and x_3. The variable x_4 is a free variable and the reduced form of the system is

$$\begin{cases} x_1 = 7x_4 + 8 \\ x_2 = -3x_4 + 2 \\ x_3 = -x_4 - 5. \end{cases}$$

Each solution of the system can be written as

$$x_1 = 7t + 8 \quad x_2 = -3t + 2 \quad x_3 = -t - 5 \quad x_4 = t,$$

where t is a real number.

(b) The leading variables are x_1 and x_3. The variable x_2 is a free variable. In view of the third row of the matrix which corresponds to the equation $0 = 1$, the system has no solutions. It doesn't make sense to write down the reduced form of the system.

2.8 (a) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 2 & -3 & -2 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1^* & -\frac{3}{2} & -1 \\ 2 & 1 & 1 \\ 3 & 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1^* & -\frac{3}{2} & -1 \\ 0 & 4 & 3 \\ 0 & \frac{13}{2} & 4 \end{bmatrix}$$
$$\to \begin{bmatrix} 1^* & -\frac{3}{2} & -1 \\ 0 & 1^* & \frac{3}{4} \\ 0 & \frac{13}{2} & 4 \end{bmatrix} \to \begin{bmatrix} 1^* & 0 & \frac{1}{8} \\ 0 & 1^* & \frac{3}{4} \\ 0 & 0 & -\frac{7}{8} \end{bmatrix}.$$

In view of the last row of the reduced matrix the system is inconsistent.

(c) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} -2 & 1 & -3 & 1 \\ 5 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1^* & -\frac{1}{2} & \frac{3}{2} & -\frac{1}{2} \\ 5 & -2 & 6 & 0 \end{bmatrix} \to \begin{bmatrix} 1^* & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & \frac{1}{2} & -\frac{3}{2} & \frac{5}{2} \end{bmatrix}$$
$$\to \begin{bmatrix} 1^* & -\frac{1}{2} & \frac{3}{2} & \frac{1}{2} \\ 0 & 1^* & -3 & 5 \end{bmatrix} \to \begin{bmatrix} 1^* & 0 & 0 & 3 \\ 0 & 1^* & -3 & 5 \end{bmatrix}.$$

The basic variables are x_1 and x_2. The free variable is x_3. The reduced form of the system is

$$\begin{cases} x_1 = 3 \\ x_2 = 5 + 3s, \end{cases}$$

where $s \in \mathbb{R}$.

Each solution of the system can be written as

$$x_1 = 3 \quad x_2 = 5 + 3s \quad x_3 = s,$$

where s is a real number.

(d) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 0 & 0 & 1 & 2 & -1 & 4 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 2 & 4 & 1 & 7 & 0 & 7 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_4} \begin{bmatrix} 2 & 4 & 1 & 7 & 0 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{r_1/2} \begin{bmatrix} 1^* & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 0 & 1 & -1 & 3 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 1 & 2 & -1 & 4 \end{bmatrix}$$

$$\xrightarrow{r_2 \leftrightarrow r_4} \begin{bmatrix} 1^* & 2 & \frac{1}{2} & \frac{7}{2} & 0 & \frac{7}{2} \\ 0 & 0 & 1^* & 2 & -1 & 4 \\ 0 & 0 & 1 & 3 & -2 & 7 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow[r_3 - r_2]{r_1 - \frac{1}{2}r_2} \begin{bmatrix} 1^* & 2 & 0 & \frac{5}{2} & \frac{1}{2} & \frac{3}{2} \\ 0 & 0 & 1^* & 2 & -1 & 4 \\ 0 & 0 & 0 & 1^* & -1 & 3 \\ 0 & 0 & 0 & 1 & -1 & 3 \end{bmatrix}$$

$$\xrightarrow[\substack{r_2 - 2r_3 \\ r_4 - r_3}]{r_1 - \frac{5}{2}r_3} \begin{bmatrix} 1^* & 2 & 0 & 0 & 3 & -6 \\ 0 & 0 & 1^* & 0 & 1 & -2 \\ 0 & 0 & 0 & 1^* & -1 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Hence the basic variables are u, w and x, while v and y are the free variables.

The reduced form of the system is

$$\begin{cases} u = -2s - 3t - 6 \\ w = -t - 2 \\ x = t + 3, \end{cases}$$

where $s, t \in \mathbb{R}$.

Each solution of the system can be written as

$$u = -2s - 3t - 6 \quad v = s \quad w = -t - 2 \quad x = t + 3 \quad y = t,$$

where s and t are real numbers.

2.9 (a) Reduction of the coefficient matrix leads to

$$\begin{bmatrix} 2 & -1 & -3 \\ -1 & 2 & -3 \\ 1 & 1 & 4 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_3} \begin{bmatrix} 1^* & 1 & 4 \\ -1 & 2 & -3 \\ 2 & -1 & -3 \end{bmatrix} \xrightarrow[r_3 - 2r_1]{r_2 + r_1} \begin{bmatrix} 1^* & 1 & 4 \\ 0 & 3 & 1 \\ 0 & -3 & -11 \end{bmatrix}$$

$$\xrightarrow{r_2/3} \begin{bmatrix} 1^* & 1 & 4 \\ 0 & 1^* & \frac{1}{3} \\ 0 & -3 & -11 \end{bmatrix} \xrightarrow[r_3 + 3r_2]{r_1 - r_2} \begin{bmatrix} 1^* & 0 & \frac{11}{3} \\ 0 & 1^* & \frac{1}{3} \\ 0 & 0 & -10 \end{bmatrix} \xrightarrow[\substack{r_1 - \frac{11}{3}r_3 \\ r_2 - \frac{1}{3}r_3}]{-\frac{1}{10}r_3} \begin{bmatrix} 1^* & 0 & 0 \\ 0 & 1^* & 0 \\ 0 & 0 & 1^* \end{bmatrix}.$$

The unique solution of the system is $x = 0, y = 0, z = 0$.

(b) Reduction of the coefficient matrix leads to

$$\begin{bmatrix} 0 & 1 & 3 & -2 \\ 2 & 1 & -4 & 3 \\ 2 & 3 & 2 & -1 \\ -4 & -3 & 5 & -4 \end{bmatrix} \xrightarrow{r_1 \leftrightarrow r_2} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 2 & 3 & 2 & -1 \\ -4 & -3 & 5 & -4 \end{bmatrix} \xrightarrow[r_4 - 2r_1]{r_3 - r_1} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 2 & 6 & -4 \\ 0 & -1 & -3 & 2 \end{bmatrix}$$

$$\xrightarrow[r_4 + r_2]{r_3 - 2r_2} \begin{bmatrix} 2 & 1 & -4 & 3 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \xrightarrow{r_1 - r_2} \begin{bmatrix} 2 & 0 & -7 & 5 \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\xrightarrow{\frac{1}{2}r_1} \begin{bmatrix} 1 & 0 & -\frac{7}{2} & \frac{5}{2} \\ 0 & 1 & 3 & -2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

The basic variables are u and v. The free variables are w and x.

The reduced form of the system is

$$\begin{cases} u = \frac{7}{2}s - \frac{5}{2}t \\ v = -3s + 2t, \end{cases}$$

where $s, t \in \mathbb{R}$.

Each solution of the system can be written as

$$u = \tfrac{7}{2}s - \tfrac{5}{2}t \quad v = -3s + 2t \quad w = s \quad x = t,$$

where s and t are real numbers.

2.10 (a) The three lines do not coincide and they pass through the origin.

(b) The three lines coincide.

2.11 (a) Since $ad - bc \neq 0$, we may conclude that $a \neq 0$ or $c \neq 0$; say $a \neq 0$.

Then reduction of the matrix leads to

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\frac{r_1}{a}} \begin{bmatrix} 1^* & \frac{b}{a} \\ c & d \end{bmatrix} \xrightarrow{r_2 - cr_1} \begin{bmatrix} 1^* & \frac{b}{a} \\ 0 & d - \frac{bc}{a} \end{bmatrix} = \begin{bmatrix} 1^* & \frac{b}{a} \\ 0 & \frac{ad - bc}{a} \end{bmatrix}$$

$$\xrightarrow{r_2 / \frac{ad - bc}{a}} \begin{bmatrix} 1^* & \frac{b}{a} \\ 0 & 1^* \end{bmatrix} \xrightarrow{r_1 - \frac{b}{a}r_2} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

If $c \neq 0$, then reduction leads to a similar result.

(b) According to part (a) reduction of the matrix

$$\begin{bmatrix} a & b & k \\ c & d & l \end{bmatrix}$$

leads to a matrix of the form

$$\begin{bmatrix} 1 & 0 & * \\ 0 & 1 & * \end{bmatrix}.$$

This proves that the given system has a unique solution.

2.13 (a) If $x = s_1, y = t_1$ and $x = s_2, y = t_2$ are solutions of (1), then

$$\begin{cases} as_1 + bt_1 = k \\ cs_1 + dt_1 = l \end{cases} \quad \text{and} \quad \begin{cases} as_2 + bt_2 = k \\ cs_2 + dt_2 = l. \end{cases}$$

Then however

$$a(s_1 - s_2) + b(t_1 - t_2) = as_1 - as_2 + bt_1 - bt_2 = as_1 + bt_1 - (as_2 + bt_2) = k - k = 0.$$

Similarly,

$$c(s_1 - s_2) + d(t_1 - t_2) = \ldots = l - l = 0.$$

Hence, $(x = s_1 - s_2, y = t_1 - t_2)$ is a solution of (2).

2.14 A reduced $n \times n$ matrix containing no zero row has the form

$$\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

2.15 (a) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 3 & 1 & 2 & p \\ 0 & q & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & p-6 \\ 0 & q & 8 & 8 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & p-6 \\ 0 & 0 & 8+q & 8-pq+6q \end{bmatrix}.$$

It is possible that the system is inconsistent only if $q = -8$. If we substitute this value of q in the foregoing matrix, we get

$$\begin{bmatrix} 1 & 0 & 1 & 2 \\ 0 & 1 & -1 & p-6 \\ 0 & 0 & 0 & 8p-40 \end{bmatrix}.$$

Hence the system is inconsistent if $q = -8$ and $p \neq 5$.

(b) If $q \neq -8$ the system has a unique solution for all values of p.