2.9 (b) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 4 & -8 & 12 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{bmatrix} \to \begin{bmatrix} 1^* & -2 & 3 \\ 3 & -6 & 9 \\ -2 & 4 & -6 \end{bmatrix} \to \begin{bmatrix} 1^* & -2 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The basic variable is x_1 and the free variable is x_2. The reduced form of the system is

$$\begin{cases} x_1 = 2t + 3 \\ x_2 = t, \end{cases}$$

where $t \in \mathbb{R}$.

2.14 Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 1^* & 2 & -3 & 4 \\ 3 & -1 & 5 & 2 \\ 4 & 1 & a^2-14 & a+2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & -7 & a^2-2 & a-14 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & -7 & 14 & -10 \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & a^2-16 & a-4 \end{bmatrix}.$$

We distinguish three cases:

If $a = 4$, then further reduction leads to

$$\to \begin{bmatrix} 1^* & 2 & -3 & 4 \\ 0 & 1^* & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

In this case we have an infinite number of solutions.

If $a = -4$, then further reduction leads to

$$\begin{bmatrix} 1 & 2 & -3 & 4 \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 0 & -8 \end{bmatrix}.$$

In this case the system is inconsistent: there are no solutions.

If $a \neq 4$ ànd $a \neq -4$, then further reduction leads to

$$\begin{bmatrix} 1 & 0 & 1 & \frac{8}{7} \\ 0 & 1 & -2 & \frac{10}{7} \\ 0 & 0 & 1^* & \frac{1}{a+4} \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & \frac{8}{7} - \frac{1}{a+4} \\ 0 & 1 & 0 & \frac{\frac{10}{7}+2}{a+4]} \\ 0 & 0 & 1 & \frac{1}{a+4} \end{bmatrix}.$$

In this case the system has a unique solution.

2.15 (b) Since (s_1, t_1) is a solution of the system (1),

$$\begin{cases} a\,s_1 + b\,t_1 = k \\ a\,s_1 + d\,t_1 = l \end{cases}$$

and since (s_0, t_0) is a solution of the system (2)

$$\begin{cases} a\,s_0 + b\,t_0 = 0 \\ c\,s_0 + d\,t_0 = 0. \end{cases}$$

So

$$a(s_1+s_0)+b(t_1+t_0)=as_1+as_0+bt_1+bt_0=as_1+bt_1+as_0+bt_0=k+0=k.$$

Similarly,

$$c(s_1+s_0)+d(t_1+t_0)=\ldots=l+0=l.$$

Hence (s_1+s_0,t_1+t_0) is a solution of the system (1).

2.18 We distinguish two cases.

If $c=0$, then the augmented coefficient matrix is

$$\begin{bmatrix}1&0&2&2\\0&0&0&1\\0&0&0&d\end{bmatrix}.$$

In view of the second row of this matrix, the system is inconsistent for any value of d.

If $c\neq 0$, reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix}1&0&2&2\\0&c&c&1\\0&0&cd&c+d\end{bmatrix}\to\begin{bmatrix}1&0&2&2\\0&1&1&c^{-1}\\0&0&cd&c+d\end{bmatrix}.$$

If $d=0$, according to the third row of this matrix, the system is inconsistent for any value of $c\neq 0$.

If $d\neq 0$, further reduction of the matrix leads to

$$\begin{bmatrix}1&0&2&2\\0&1&1&c^{-1}\\0&0&1&\dfrac{c+d}{cd}\end{bmatrix}\to\begin{bmatrix}1&0&0&2-2\dfrac{c+d}{cd}\\0&1&0&c^{-1}-\dfrac{c+d}{cd}\\0&0&1&\dfrac{c+d}{cd}\end{bmatrix}.$$

Hence, the system has a unique solution if $c\neq 0$ and $d\neq 0$.

As we observed before, the system is inconsistent if (1) $c=0$ or (2) $d=0$ and $c\neq 0$.

2.19 (a) Note that $\underline{x}$ is in the intersection of the three planes if and only if

$$\begin{cases}\underline{x}\cdot\underline{n}_1=2\\\underline{x}\cdot\underline{n}_2=0\\\underline{x}\cdot\underline{n}_3=q\end{cases}\iff\begin{bmatrix}1&0&4\\0&2&4\\1&p&p^2q\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}2\\0\\q\end{bmatrix}.$$

So we should in fact determine those values of p and q such that the foregoing system of linear equations has an infinite number of solutions.

In order to answer this question, we reduce the augmented coefficient matrix of the system

$$\begin{bmatrix}1&0&4&2\\0&2&4&0\\1&p&p^2q&q\end{bmatrix}\to\begin{bmatrix}1&0&4&2\\0&1&2&0\\0&p&p^2q-4&q-2\end{bmatrix}.$$

If $p=0$, the system has a unique solution. If $p\neq 0$, we further reduce the matrix to

$$\begin{bmatrix}1&0&4&2\\0&1&2&0\\0&0&p^2q-2p-4&q-2\end{bmatrix}.$$

The system has an infinite number of solutions if (and only if) $p^2q - 2p - 4 = 0$ and $q - 2 = 0$. So $q = 2$ and $p^2 - p - 2 = 0$. Hence, $p = -1$ or $p = 2$.

(b) For the values of p and q we found in part (a), the reduced matrix is

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Hence,

$$\begin{cases} x = 2 - 4z \\ y = -2z \end{cases}$$

is the reduced form of the system and each solution is of the form

$$\begin{bmatrix} 2 - 4z \\ -2z \\ z \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix},$$

where $z \in \mathbb{R}$. So the intersection of the three planes is a line with vector representation

$$\underline{v} = \begin{bmatrix} 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -4 \\ -2 \\ 1 \end{bmatrix}.$$

(c) In fact we should determine those values of p such that the foregoing system of linear equations has no solutions.

In order to answer this question, we reduce the augmented coefficient matrix of the system

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 2 & 4 & 0 \\ 1 & p & p^2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & p & p^2 - 4 & -1 \end{bmatrix}.$$

Obviously, if $p = 0$, the system has a unique solution.

If $p \neq 0$, further reduction leads to

$$\begin{bmatrix} 1 & 0 & 4 & 2 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & p^2 - 2p - 4 & -1 \end{bmatrix}.$$

The system has no solutions if

$$p^2 - 2p - 4 = 0 \Longleftrightarrow p = \frac{2 \pm \sqrt{4 + 16}}{2} = 1 \pm \sqrt{5}.$$

(d) If $\underline{v}$ is a point contained in the three planes with $v_1 = 1$, then v_2 and v_3 satisfy

$$\begin{cases} 1 + 4v_3 = 2 \\ 2v_2 + 4v_3 = 0 \\ 1 + pv_2 + p^2 v_3 = 1. \end{cases}$$

According to the first equation, $v_3 = \frac{1}{4}$. In combination with the second equation this leads to $2v_2 + 1 = 0 \Longrightarrow v_2 = -\frac{1}{2}$. Hence,

$$1 + pv_2 + p^2 v_3 = 1 \Longrightarrow 1 - \tfrac{1}{2}p + \tfrac{1}{4}p^2 = 1 \Longrightarrow p^2 - 2p = 0 \Longrightarrow p = 0 \quad \text{or} \quad p = 2.$$