3.1 (a) Note that

$$\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} + \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} \underset{\text{def}}{=} \begin{bmatrix} u_1 + v_1 \\ \vdots \\ u_n + v_n \end{bmatrix} \underset{\substack{\text{property of} \\ \text{real numnbers}}}{=} \begin{bmatrix} v_1 + u_1 \\ \vdots \\ v_n + u_n \end{bmatrix} \underset{\text{def}}{=} \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}.$$

(b) Note that

$$c(d\,\underline{u}) = c\left(d\begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix}\right) \underset{\text{def}}{=} c\begin{bmatrix} du_1 \\ \vdots \\ du_n \end{bmatrix} \underset{\text{def}}{=} \begin{bmatrix} cdu_1 \\ \vdots \\ cdu_n \end{bmatrix} \underset{\substack{\text{property of} \\ \text{real numnbers}}}{=} \begin{bmatrix} dcu_1 \\ \vdots \\ dcu_n \end{bmatrix} \underset{\text{def}}{=} d\begin{bmatrix} cu_1 \\ \vdots \\ cu_n \end{bmatrix} \underset{\text{def}}{=} d(c\,\underline{u}).$$

3.2 The vectors are

$$\text{(a)} \begin{bmatrix} -1 \\ 9 \\ -11 \\ 1 \end{bmatrix} \quad \text{(b)} \begin{bmatrix} 22 \\ 53 \\ -19 \\ 14 \end{bmatrix} \quad \text{(c)} \begin{bmatrix} -13 \\ 13 \\ -36 \\ -2 \end{bmatrix} \quad \text{(d)} \begin{bmatrix} -90 \\ -114 \\ 60 \\ -36 \end{bmatrix} \quad \text{(e)} \begin{bmatrix} -9 \\ -5 \\ -5 \\ -3 \end{bmatrix} \quad \text{(f)} \begin{bmatrix} 27 \\ 29 \\ -27 \\ 9 \end{bmatrix}.$$

3.3 (a) Note that $5\,\underline{x} - 2\,\underline{v} = 2\,\underline{w} - 10\underline{x}$. So

$$\underline{x} = \tfrac{2}{15}(\underline{v} + \underline{w}) = \tfrac{2}{15}\begin{bmatrix} 9 \\ 5 \\ 5 \\ 3 \end{bmatrix} = \begin{bmatrix} \frac{6}{5} \\ \frac{2}{3} \\ \frac{2}{3} \\ \frac{2}{5} \end{bmatrix}.$$

(b) We have to find numbers x_1 and x_2 such that $\underline{u} = x_1\,\underline{v} + x_2\,\underline{w}$.

Therefore we reduce the following (augmented) coefficient matrix:

$$\begin{bmatrix} 4 & 5 & -3 \\ 7 & -2 & 2 \\ -3 & 8 & 1 \\ 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 1 & 0 \\ 4 & 5 & -3 \\ 7 & -2 & 2 \\ -3 & 8 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & 0 \\ 0 & 3 & -3 \\ 0 & -5\frac{1}{2} & 2 \\ 0 & 9\frac{1}{2} & 1 \end{bmatrix} \to \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & -1 \\ 0 & 0 & -3\frac{1}{2} \\ 0 & 0 & 10\frac{1}{2} \end{bmatrix}.$$

In view of the last two rows of the reduced matrix, $\underline{u}$ is no linear combination of $\underline{v}$ and $\underline{w}$.

3.4 Obviously, $\underline{a} = -\frac{5}{4}\underline{b} - \frac{3}{2}\underline{c}$.

The fact that the vectors $\underline{a}, \underline{b}$ and $\underline{c}$ are an element of $\mathbb{R}^8$ is irrelevant.

3.5 Obviously, $\underline{x} = \underline{u} + \frac{2}{3}\underline{v} - 2\,\underline{w}$.

3.7 We proved that each vector in $\mathbb{R}^n$ can be written as a linear combination of the unit vectors in $\mathbb{R}^n$.

Hence, $\mathbb{R}^n \subset \text{span}\{\underline{e}_1, \ldots, \underline{e}_n\}$.

Since the other inclusion is obvious, the proof is complete.

3.10 Note that

$$\begin{aligned} \tfrac{1}{4}\|\underline{u} + \underline{v}\|^2 - \tfrac{1}{4}\|\underline{u} - \underline{v}\|^2 &= \tfrac{1}{4}(\underline{u} + \underline{v}) \cdot (\underline{u} + \underline{v}) - \tfrac{1}{4}(\underline{u} - \underline{v}) \cdot (\underline{u} - \underline{v}) \\ &= \tfrac{1}{4}(\underline{u} \cdot \underline{u}) + \tfrac{1}{2}(\underline{u} \cdot \underline{v}) + \tfrac{1}{4}(\underline{v} \cdot \underline{v}) - \tfrac{1}{4}(\underline{u} \cdot \underline{u}) + \tfrac{1}{2}(\underline{u} \cdot \underline{v}) - \tfrac{1}{4}(\underline{v} \cdot \underline{v}) \\ &= \underline{u} \cdot \underline{v}. \end{aligned}$$

3.11 Note that

$$\|\underline{u}+\underline{v}\|^2+\|\underline{u}-\underline{v}\|^2 = (\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})+(\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v})$$
$$= \|\underline{u}\|^2+2(\underline{u}\cdot\underline{v})+\|\underline{v}\|^2+\|\underline{u}\|^2-2(\underline{u}\cdot\underline{v})+\|\underline{v}\|^2$$
$$= 2\|\underline{u}\|^2+2\|\underline{v}\|^2.$$

3.12 (a) Note that

$$(\underline{u}+\underline{v})\cdot(\underline{v}+\underline{w}) = \underline{u}\cdot\underline{v}+\underline{u}\cdot\underline{w}+\underline{v}\cdot\underline{v}+\underline{v}\cdot\underline{w} = 2+5+4-3 = 8.$$

(b) Note that

$$(2\,\underline{v}-\underline{w})\cdot(3\,\underline{u}+2\,\underline{w}) = 6(\underline{v}\cdot\underline{u})+4(\underline{v}\cdot\underline{w})-3(\underline{w}\cdot\underline{u})-2(\underline{w}\cdot\underline{w})$$
$$= 6(\underline{u}\cdot\underline{v})+4(\underline{v}\cdot\underline{w})-3(\underline{u}\cdot\underline{w})-2\|\underline{w}\|^2 = 12-12-15-98 = -113.$$

(c) Note that

$$\langle\underline{u}-\underline{v}-2\,\underline{w},4\,\underline{u}+\underline{v}\rangle = 4\langle\underline{u},\underline{u}\rangle+\langle\underline{u},\underline{v}\rangle-4\langle\underline{u},\underline{v}\rangle-\langle\underline{v},\underline{v}\rangle-8\langle\underline{u},\underline{w}\rangle-2\langle\underline{v},\underline{w}\rangle$$
$$= 4\|\underline{u}\|^2-3\langle\underline{u},\underline{v}\rangle-\|\underline{v}\|^2-8\langle\underline{u},\underline{w}\rangle-2\langle\underline{v},\underline{w}\rangle$$
$$= 4-6-4-40+6 = -40.$$

(d) Note that

$$\|\underline{u}+\underline{v}\| = \sqrt{(\underline{u}+\underline{v})\cdot(\underline{u}+\underline{v})} = \sqrt{\|\underline{u}\|^2+2(\underline{u}\cdot\underline{v})+\|\underline{v}\|^2} = \sqrt{1+4+4} = 3.$$

(e) Note that

$$\|2\,\underline{w}-\underline{v}\| = \sqrt{(2\,\underline{w}-\underline{v})\cdot(2\,\underline{w}-\underline{v})} = \sqrt{4\|\underline{w}\|^2-4(\underline{v}\cdot\underline{w})+\|\underline{v}\|^2} = \sqrt{196+12+4} = \sqrt{212} = 2\sqrt{53}.$$

(f) Note that

$$\|\underline{u}-2\,\underline{v}+4\,\underline{w}\| = \sqrt{1+16+16\cdot 49+(-8)+40+48} = \sqrt{881}.$$

3.15 (a) The two vectors are orthogonal because their dot product is equal to

$$-4+6-2 = 0.$$

(b) The two vectors are orthogonal because the vector $\underline{0}$ is orthogonal to each vector.

(c) The dot product of the two vectors is $-2ab$. Hence, the vectors are orthogonal only if $a = 0$ or $b = 0$.

3.16 (a) Note that

$$\underline{u}\cdot\underline{v} = 0 \iff k^2+k+k = 0 \iff k = 0 \quad\text{or}\quad k = -2.$$

(b) Note that

$$\frac{\underline{u}\cdot\underline{v}}{\|\underline{u}\|\|\underline{v}\|} = \cos\tfrac{\pi}{3} \iff \frac{k^2+2k}{\sqrt{(2k^2+1)(2k^2+1)}} = \tfrac{1}{2} \iff k^2+2k = \tfrac{1}{2}(2k^2+1) \iff k = \tfrac{1}{4}.$$