3.6 If $\underline{v}$ is a multiple of $\underline{u}$, then $\underline{v} = t\,\underline{u}$ for some $t \in \mathbb{R}$.

In that situation $\text{span}\{\underline{u}, \underline{v}\} = \text{span}\{\underline{u}\}$, which implies that the span of $\underline{u}$ and $\underline{v}$ is the line through the origin and the point (u_1, u_2, u_3).

Obviously, $\text{span}\{\underline{u}\} \subset \text{span}\{\underline{u}, \underline{v}\}$. To prove the reverse inclusion, assume that $\underline{w} \in \text{span}\{\underline{u}, \underline{v}\}$. Then numbers c and d exist such that

$$\underline{w} = c\,\underline{u} + d\,\underline{v} = c\,\underline{u} + d(t\,\underline{u}) = (c + dt)\,\underline{u}.$$

So $\underline{w}$ is a multiple of $\underline{u}$, that is: $\underline{w} \in \text{span}\{\underline{u}\}$. By consequence, $\text{span}\{\underline{u}, \underline{v}\} \subset \text{span}\{\underline{u}\}$.

3.8 (a) By choosing $c_1 = \cdots = c_m = 0$ it follows that

$$\underline{0} = c_1\,\underline{u}_1 + \cdots + c_m\,\underline{u}_m \in \text{span}\{\underline{u}_1, \ldots, \underline{u}_m\} = V.$$

So the set V is nonempty.

(b) If $\underline{v} \in V$, there exist numbers $c_1, \ldots, c_m$ such that

$$\underline{v} = c_1\,\underline{u}_1 + \cdots + c_m\,\underline{u}_m.$$

Similarly, numbers $d_1, \ldots, d_m$ exist such that

$$\underline{w} = d_1\,\underline{u}_1 + \cdots + d_m\,\underline{u}_m.$$

Then, according to Theorem 1,

$$\begin{aligned} \underline{v} + \underline{w} &= \big(c_1\,\underline{u}_1 + \cdots + c_m\,\underline{u}_m\big) + \big(d_1\,\underline{u}_1 + \cdots + d_m\,\underline{u}_m\big) \\ &= (c_1 + d_1)\,\underline{u}_1 + \cdots + (c_m + d_m)\,\underline{u}_m \in \text{span}\{\underline{u}_1, \ldots, \underline{u}_m\}. \end{aligned}$$

(c) As $\underline{v} \in V$, there exist numbers $d_1, \ldots, d_m$ such that

$$\underline{v} = d_1\,\underline{u}_1 + \cdots + d_m\,\underline{u}_m.$$

Then, according to Theorem 1,

$$c\,\underline{v} = c\big(d_1\,\underline{u}_1 + \cdots + d_m\,\underline{u}_m\big) = (cd_1)\,\underline{u}_1 + \cdots + (cd_m)\,\underline{u}_m \in \text{span}\{\underline{u}_1, \ldots, \underline{u}_m\}.$$

3.9 Note that

$$\|\underline{u} + \underline{v}\| \geq \big|\|\underline{u}\| - \|\underline{v}\|\big| \Longleftrightarrow -\|\underline{u} + \underline{v}\| \leq \|\underline{u}\| - \|\underline{v}\| \leq \|\underline{u} + \underline{v}\|.$$

Now (use the Triangle Inequality)

$$\|\underline{v}\| = \|\underline{v} + \underline{u} - \underline{u}\| \leq \|\underline{v} + \underline{u}\| + \| - \underline{u}\| = \|\underline{u} + \underline{v}\| + \|\underline{u}\|$$

implies that $\|\underline{u}\| - \|\underline{v}\| \geq -\|\underline{u} + \underline{v}\|$. Further, (again use the Triangle Inequality)

$$\|\underline{u}\| = \|\underline{u} + \underline{v} - \underline{v}\| \leq \|\underline{u} + \underline{v}\| + \| - \underline{v}\| = \|\underline{u} + \underline{v}\| + \|\underline{v}\|$$

implies that $\|\underline{u}\| - \|\underline{v}\| \leq \|\underline{u} + \underline{v}\|$.

3.13 With the help of the Triangle Inequality we find that for any vector $\underline{w}$

$$\|\underline{u}-\underline{v}\| = \|\underline{u}-\underline{w}+\underline{w}-\underline{v}\| \leq \|\underline{u}-\underline{w}\| + \|\underline{w}-\underline{v}\|.$$

3.14 Note that

$$\begin{aligned}\|\underline{u}-\underline{v}\| = \|\underline{u}+\underline{v}\| &\Longleftrightarrow \|\underline{u}-\underline{v}\|^2 = \|\underline{u}+\underline{v}\|^2 \\ &\Longleftrightarrow \|\underline{u}\|^2 - 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2 = \|\underline{u}\|^2 + 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2 \\ &\Longleftrightarrow 4(\underline{u}\cdot\underline{v}) = 0 \\ &\Longleftrightarrow \underline{u}\cdot\underline{v} = 0.\end{aligned}$$

3.17 (a) Obviously,

$$(\underline{u}-\widehat{\underline{u}})\cdot\underline{v} = \underline{u}\cdot\underline{v} - \widehat{\underline{u}}\cdot\underline{v} = \underline{u}\cdot\underline{v} - \frac{\underline{u}\cdot\underline{v}}{\|\underline{v}\|^2}(\underline{v}\cdot\underline{v}) = \underline{u}\cdot\underline{v} - \underline{u}\cdot\underline{v} = 0.$$

(b) The vector $\widehat{\underline{u}}$ is the orthogonal projection of the vector $\underline{u}$ on the vector $\underline{v}$.

3.18 For orthonormal vectors $\underline{u}$ and $\underline{v}$,

$$\|\underline{u}-\underline{v}\|^2 = (\underline{u}-\underline{v})\cdot(\underline{u}-\underline{v}) = \|\underline{u}\|^2 - 2(\underline{u}\cdot\underline{v}) + \|\underline{v}\|^2 = 1-0+1 = 2.$$

So $\|\underline{u}-\underline{v}\| = \sqrt{2}$.

3.19 If the vectors $\underline{u}$ and $\underline{v}$ are orthogonal, then

$$\underline{u}\cdot\underline{v} = 0 \Longrightarrow -1+q+2p = 0.$$

Furthermore, $\underline{w}$ is a linear combination of $\underline{u}$ and $\underline{v}$. So for some numbers a and b, $\underline{w} = a\,\underline{u} + b\,\underline{v}$. Hence,

$$\begin{cases} a-b=2 \\ a+bq=3 \\ ap+2b=1 \end{cases} \Longrightarrow \begin{cases} b+2+bq=3 \\ (b+2)p+2b=1 \end{cases} \Longrightarrow \begin{cases} q = \dfrac{1-b}{b} \\ p = \dfrac{1-2b}{b+2}. \end{cases}$$

In combination with the first equation, this leads to

$$\begin{aligned}-1+\frac{1-b}{b}+\frac{2-4b}{b+2} = 0 &\Longrightarrow -b(b+2)+(1-b)(b+2)+(2-4b)b = 0 \\ &\Longrightarrow -b^2-2b+b+2-b^2-2b+2b-4b^2 = 0 \Longrightarrow -6b^2-b+2 = 0 \\ &\Longrightarrow b = \frac{1\pm\sqrt{1+48}}{-12} \Longrightarrow b = -\tfrac{2}{3} \text{ or } b = \tfrac{1}{2}.\end{aligned}$$

Note that the cases $b=0$ and $b=-2$ do not lead to a solution.
If $b = -\frac{2}{3}$, then $p = \frac{7}{4}$ and $q = -\frac{5}{2}$.
If $b = \frac{1}{2}$, then $p = 0$ and $q = 1$.