4.3 The second equation is $0 = -2$. So no scalars c_1, c_2 and c_3 exist satisfying the given equation.

4.4 We have to find numbers c_1 and c_2 such that $\underline{b} = c_1\,\underline{a}_1 + c_2\,\underline{a}_2$, where $\underline{a}_i$ represents the ith column of the matrix A.

Therefore we reduce the following (augmented) coefficient matrix:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 7 & -3 \\ -3 & -9 & 8 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & -3 & 5 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 2 & -1 \\ 0 & 3 & -1 \\ 0 & 0 & 4 \end{bmatrix}.$$

Due to the last row of this matrix such numbers don't exist.

4.6 Obviously $2\,\underline{u} - 5\,\underline{v} = \underline{w}$.

So $x_1 = 2, x_2 = -5$ is a solution of the equation.

In order to find all the solutions we reduce the augmented coefficient matrix of the system:

$$\begin{bmatrix} 3 & 1 & 1 \\ 8 & 3 & 1 \\ 4 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & \frac{1}{3} & \frac{1}{3} \\ 0 & \frac{1}{3} & -\frac{5}{3} \\ 0 & -\frac{1}{3} & \frac{5}{3} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -5 \\ 0 & 0 & 0 \end{bmatrix}.$$

So the solution we have found is the only one.

4.7 (a) Obviously,

$$A(\underline{u} + \underline{v}) = \begin{bmatrix} 5 & 1 & -3 \\ 7 & -2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \end{bmatrix}.$$

Note that

$$A\underline{u} + A\underline{v} = \begin{bmatrix} -3 \\ 8 \end{bmatrix} + \begin{bmatrix} -9 \\ -16 \end{bmatrix} = \begin{bmatrix} -12 \\ -8 \end{bmatrix}.$$

(b) Obviously,

$$A(5\,\underline{u}) = \begin{bmatrix} 5 & 1 & -3 \\ 7 & -2 & 1 \end{bmatrix} \begin{bmatrix} 5 \\ 5 \\ 15 \end{bmatrix} = \begin{bmatrix} -15 \\ 40 \end{bmatrix}$$

and

$$5(A\underline{u}) = 5 \begin{bmatrix} -3 \\ 8 \end{bmatrix} = \begin{bmatrix} -15 \\ 40 \end{bmatrix}.$$

4.8 (a) The matrix

$$A = \begin{bmatrix} 2 & -3 & 5 & 7 \\ 9 & -1 & 1 & -1 \\ 1 & 5 & 4 & 0 \end{bmatrix}$$

is the 3×3 coefficient matrix of the system, $\underline{b} = \begin{bmatrix} 7 \\ -1 \\ 0 \end{bmatrix}$ is the known term and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is the vector of unknowns.

(b) The matrix

$$A = \begin{bmatrix} 4 & 0 & -3 & 1 & 1 \\ 5 & 1 & 0 & -8 & 3 \\ 2 & -5 & 9 & -1 & 0 \\ 0 & 3 & -1 & 7 & 2 \end{bmatrix}$$

is the 4×4 coefficient matrix of the system, $\underline{b} = \begin{bmatrix} 1 \\ 3 \\ 0 \\ 2 \end{bmatrix}$ is the known term and $\underline{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$ is the vector of unknowns.

4.9 (a) A system is

$$\begin{cases} 3x_1 - x_2 + 2x_3 = 2 \\ 4x_1 + 3x_2 + 7x_3 = -1 \\ -2x_1 + x_2 + 5x_3 = 4. \end{cases}$$

(b) A system is

$$\begin{cases} 3w - 2x + z = 0 \\ 5w + 2y - 2z = 0 \\ 3w + x + 4y + 7z = 0 \\ -2w + 5x + y + 6z = 0. \end{cases}$$

4.10 (a) Reduction of the coefficient matrix leads to

$$\begin{bmatrix} 2 & -3 \\ 2 & 1 \\ 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & -3 \\ 0 & 4 \\ 3 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 0 \\ 0 & 1 \\ 3 & 0 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

So the solution set is

$$\left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\}.$$

4.12 (a) The solution set is

$$\left\{ \begin{bmatrix} -3 \\ 0 \\ 7 \end{bmatrix} \right\}.$$

(b) The free variable is x_4, while the reduced form of the system is given by

$$\begin{cases} x_1 = 8 + 7x_4 \\ x_2 = 2 - 3x_4 \\ x_3 = -5 - x_4. \end{cases}$$

Each solution of the system has the form

$$\begin{bmatrix} 8+7t \\ 2-3t \\ -5-t \\ t \end{bmatrix} = \begin{bmatrix} 8 \\ 2 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ -1 \\ 1 \end{bmatrix},$$

where $t \in \mathbb{R}$.

So the solution set is given by

$$\left\{ \begin{bmatrix} 8 \\ 2 \\ -5 \\ 0 \end{bmatrix} + t \begin{bmatrix} 7 \\ -3 \\ -1 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

(c) The free variables are x_2 and x_5 and the reduced form of the system is

$$\begin{cases} x_1 = -2 + 6x_2 - 3x_5 \\ x_3 = 7 - 4x_5 \\ x_4 = 8 - 5x_5. \end{cases}$$

So each solution of the system has the form

$$\begin{bmatrix} -2 + 6s - 3t \\ s \\ 7 - 4t \\ 8 - 5t \\ t \end{bmatrix},$$

where $s, t \in \mathbb{R}$. So the solution set is given by

$$\left\{ \begin{bmatrix} -2 \\ 0 \\ 7 \\ 8 \\ 0 \end{bmatrix} + s \begin{bmatrix} 6 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ -4 \\ -5 \\ 1 \end{bmatrix} \mid s, t \in \mathbb{R} \right\}.$$

(d) According to the last row of the augmented coefficient matrix the system is inconsistent.

4.13 (b) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 1 & 3 & 7 & 2 & 2 \\ 1 & -12 & -11 & -16 & 5 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & -10 & -12 & -12 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & 1 & -4 & 1 \\ 0 & 5 & 6 & 6 & 1 \\ 0 & 0 & 0 & 0 & 6 \end{bmatrix}.$$

According to the last row of this matrix the system has no solutions.

4.14 (a) Since we have found the solution sets of the corresponding homogeneous systems in Exercise 12, it is sufficient to find one solution of the inhomogeneous system.

The equations 2 and 3 lead to $x_1 = 1$ and $x_2 = -1$. Substitution of these values for x_1 and x_2 in the first equation shows that the system has no solution.

4.15 In order to verify for which vectors $\underline{b}$ the system $A\underline{x} = \underline{b}$ is solvable we reduce the augmented coefficient matrix of this system:

$$\begin{bmatrix} 1 & 1 & b_1 \\ 2 & -1 & b_2 \\ 0 & 1 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & b_1 \\ 0 & -3 & b_2 - 2b_1 \\ 0 & 1 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & b_1 - b_3 \\ 0 & 1 & b_3 \\ 0 & 0 & b_2 - 2b_1 + 3b_3 \end{bmatrix}.$$

So the system $A\underline{x} = \underline{b}$ is solvable if and only if $b_2 = 2b_1 - 3b_3$. So

$$\text{Col}(A) = \{\underline{b} \in \mathbb{R}^2 | b_2 = 2b_1 - 3b_3\} = \left\{ \begin{bmatrix} b_1 \\ 2b_1 - 3b_3 \\ b_3 \end{bmatrix} \middle| b_1, b_3 \in \mathbb{R} \right\}$$

$$= \left\{ b_1 \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + b_3 \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix} \middle| b_1, b_3 \in \mathbb{R} \right\} = \text{span}\{\underline{u}, \underline{v}\},$$

where $\underline{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$ and $\underline{v} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}$.

4.16 We have to check whether the system $A\underline{x} = \underline{b}$ is solvable, where

$$A = \begin{bmatrix} 2 & 3 & -1 \\ 1 & -1 & 0 \\ 0 & 5 & 2 \\ 3 & 2 & 1 \end{bmatrix} \quad \text{and} \quad \underline{b} = \begin{bmatrix} 2 \\ 3 \\ -7 \\ 3 \end{bmatrix}.$$

In order to do so we reduce the augmented coefficient matrix of the system:

$$\begin{bmatrix} 2 & 3 & -1 & 2 \\ 1 & -1 & 0 & 3 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 2 & 3 & -1 & 2 \\ 0 & 5 & 2 & -7 \\ 3 & 2 & 1 & 3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 5 & 2 & -7 \\ 0 & 5 & 1 & -6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 0 & 3 & -3 \\ 0 & 0 & 2 & -2 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & -1 & -4 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -1 & 0 & 3 \\ 0 & 5 & 0 & -5 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since the system is solvable the vector $\underline{b}$ is contained in the span of the columns of the matrix A.

4.17 (a) We have to check whether the system $A\underline{x} = \underline{b}$ is solvable for every $\underline{b} \in \mathbb{R}^3$, where A is the matrix with the given vectors as columns.

(Partial) reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 2 & 0 & 0 & b_1 \\ 2 & 0 & 1 & b_2 \\ 2 & 3 & 1 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 \\ 0 & 0 & 1 & b_2 - b_1 \\ 0 & 3 & 1 & b_3 - b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 \\ 0 & 3 & 1 & b_3 - b_1 \\ 0 & 0 & 1 & b_2 - b_1 \end{bmatrix}$$

$$\rightarrow \begin{bmatrix} 1 & 0 & 0 & \frac{1}{2}b_1 \\ 0 & 1 & \frac{1}{3} & \frac{1}{3}b_3 - \frac{1}{3}b_1 \\ 0 & 0 & 1 & b_2 - b_1 \end{bmatrix}.$$

So the system $A\underline{x} = \underline{b}$ is solvable for every vector $\underline{b} \in \mathbb{R}^3$. Hence the given vectors span $\mathbb{R}^3$.