4.1 According to Theorem 1 part (1),

$$A(c_1\,\underline{u}_1+\cdots+c_k\,\underline{u}_k)=A(c_1\,\underline{u}_1)+\cdots+A(c_k\,\underline{u}_k).$$

According to part (2) of this theorem,

$$A(c_1\,\underline{u}_1)+\cdots+A(c_k\,\underline{u}_k)=c_1A\underline{u}_1+\cdots+c_kA\underline{u}_k.$$

4.2 We have to solve the system of equations

$$\begin{cases} -1c_1 + 2c_2 + 7c_3 + 6c_4 = 0 \\ 3c_1 + c_3 + 3c_4 = 5 \\ 2c_1 + 4c_2 + c_3 + c_4 = 6 \\ -c_2 + 4c_3 + 2c_4 = -3. \end{cases}$$

Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} -1 & 2 & 7 & 6 & 0 \\ 3 & 0 & 1 & 3 & 5 \\ 2 & 4 & 1 & 1 & 6 \\ 0 & -1 & 4 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -7 & -6 & 0 \\ 0 & 6 & 22 & 21 & 5 \\ 0 & 8 & 15 & 13 & 6 \\ 0 & -1 & 4 & 2 & -3 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -7 & -6 & 0 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 8 & 15 & 13 & 6 \\ 0 & 6 & 22 & 21 & 5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & -15 & -10 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & 47 & 29 & -18 \\ 0 & 0 & 46 & 33 & -13 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -15 & -10 & 6 \\ 0 & 1 & -4 & -2 & 3 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 46 & 33 & -13 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & -70 & -69 \\ 0 & 1 & 0 & -18 & -17 \\ 0 & 0 & 1 & -4 & -5 \\ 0 & 0 & 0 & 217 & 217 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

So $c_1=c_2=c_4=1$ and $c_3=-1$.

4.5 We have to find numbers c_1 and c_2 such that $\underline{b}=c_1\,\underline{a}_1+c_2\,\underline{a}_2$.

Therefore we reduce the following (augmented) coefficient matrix:

$$\begin{bmatrix} 1 & -5 & 3 \\ 3 & -8 & -5 \\ -1 & 2 & p \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 \\ 0 & 7 & -14 \\ 0 & -3 & p+3 \end{bmatrix} \to \begin{bmatrix} 1 & -5 & 3 \\ 0 & 1 & -2 \\ 0 & -3 & p+3 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -7 \\ 0 & 1 & -2 \\ 0 & 0 & p-3 \end{bmatrix}.$$

Due to the last row of this matrix such numbers exist only if $p=3$.

4.10 (b) Reduction of the coefficient matrix leads to

$$\begin{bmatrix} 4 & -8 \\ 3 & -6 \\ -2 & 4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence x_1 is a basic variable, while x_2 is a free variable. Furthermore the reduced form of the system is given by $x_1=2x_2$.

So the solution set is

$$\left\{ t\begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid t\in\mathbb{R} \right\}.$$

4.11 If $A\underline{u} = \underline{0}$, $A\underline{v} = \underline{0}$ and $t \in \mathbb{R}$, then according to Theorem 1,

$$A(\underline{u} + \underline{v}) = A\underline{u} + A\underline{v} = \underline{0} + \underline{0} = \underline{0}$$

and

$$A(t\,\underline{v}) = tA\underline{v} = t\,\underline{0} = \underline{0}.$$

Hence $\underline{u} + \underline{v}$ and $t\,\underline{v}$ are solutions of the system $A\underline{x} = \underline{0}$.

4.13 (a) Reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 5 & -2 & 6 & 0 \\ -2 & 1 & 3 & 1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 5 & -2 & 6 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & -\frac{1}{2} & -\frac{3}{2} & -\frac{1}{2} \\ 0 & \frac{1}{2} & 13\frac{1}{2} & 2\frac{1}{2} \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 12 & 2 \\ 0 & 1 & 27 & 5 \end{bmatrix}.$$

So x_3 is a free variable and the reduced form of the system is

$$\begin{cases} x_1 = 2 - 12t \\ x_2 = 5 - 27t, \end{cases}$$

where $t \in \mathbb{R}$. So the solution set is

$$\left\{ \begin{bmatrix} 2 \\ 5 \\ 0 \end{bmatrix} + t \begin{bmatrix} -12 \\ -27 \\ 1 \end{bmatrix} \middle| t \in \mathbb{R} \right\}.$$

4.14 (b) Since we have found the solution sets of the corresponding homogeneous systems in Exercise 10, it is sufficient to find one solution of the inhomogeneous system.

A particular solution that can be easily found is $x_1 = 1$ and $x_2 = -1$. According to Exercise 10(b) and Theorem 3 the solution set is given by

$$\left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \end{bmatrix} \mid t \in \mathbb{R} \right\}.$$

4.17 (b) We have to check whether the system $A\underline{x} = \underline{b}$ is solvable for every $\underline{b} \in \mathbb{R}^3$, where A is the matrix with the given vectors as columns.

(Partial) reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 2 & 4 & 8 & b_1 \\ -1 & 1 & -1 & b_2 \\ 3 & 2 & 8 & b_3 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & \frac{1}{2}b_1 \\ 0 & 3 & 3 & b_2 + \frac{1}{2}b_1 \\ 0 & -4 & -4 & b_3 - 1\frac{1}{2}b_1 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 4 & \frac{1}{2}b_1 \\ 0 & 1 & 1 & \frac{1}{3}b_2 + \frac{1}{6}b_1 \\ 0 & 0 & 0 & b_3 - \frac{5}{6}b_1 + \frac{4}{3}b_2 \end{bmatrix}.$$

So the system is solvable only if $b_3 - \frac{5}{6}b_1 + \frac{4}{3}b_2 = 0$, that is: the given vectors do not span $\mathbb{R}^3$.