5.1 (a) Since

$$A \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix},$$

the length of each vector is multiplied by two.

(b) $T(\underline{u}) = A\underline{u} = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 0 \\ -6 \end{bmatrix}.$

(c) Note that

$$T(\underline{x}) = \underline{v} \Longleftrightarrow \begin{bmatrix} 2x_1 \\ 2x_2 \\ 2x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -1 \\ 4 \end{bmatrix}.$$

Hence, $x_1 = \frac{5}{2}, x_2 = -\frac{1}{2}$ and $x_3 = 2$. So

$$\left\{ \begin{bmatrix} \frac{5}{2} \\ -\frac{1}{2} \\ 2 \end{bmatrix} \right\}$$

is the inverse image of $\underline{v}$.

5.2 Since

$$\begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = \begin{bmatrix} rx_1 \\ rx_2 \\ \vdots \\ rx_n \end{bmatrix} = r\,\underline{x},$$

The mapping T is the left-multiplication by the matrix $\begin{bmatrix} r & 0 & \cdots & 0 \\ 0 & r & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & r \end{bmatrix}$.

5.3 (a) In order to find those vectors $\underline{x}$ satisfying $A\underline{x} = \underline{b}$ we reduce the following matrix:

$$\begin{bmatrix} 1 & 0 & -1 & b_1 \\ 3 & 1 & -5 & b_2 \\ -4 & 2 & 0 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 2 & -4 & b_3 + 4b_1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & -1 & b_1 \\ 0 & 1 & -2 & b_2 - 3b_1 \\ 0 & 0 & 0 & b_3 - 2b_2 + 10b_1 \end{bmatrix}.$$

Apparently the system $A\underline{x} = \underline{b}$ has an infinite number of solutions if $b_3 - 2b_2 + 10b_1 = 0$.

If $b_3 - 2b_2 + 10b_1 \neq 0$, then the system has no solutions.

Hence there are no vectors such that the inverse image consists of one element only.

(b) In order to find the inverse image of $\underline{0}$ we have to solve the system $A\underline{x} = \underline{0}$.

Part (a) implies that the reduced coefficient matrix of this system is given by

$$\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix}.$$

The reduced form of the system $A\underline{x} = \underline{0}$ is

$$\begin{cases} x = z \\ y = 2z. \end{cases}$$

Hence the solution set is given by

$$\left\{ z \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} \mid z \in \mathbb{R} \right\}.$$

This set represents a line in $\mathbb{R}^3$ containing the origin.

(c) A vector $\underline{b} \in \mathbb{R}^3$ is an image for the mapping T if there exists an $\underline{x} \in \mathbb{R}^3$ such that $A\underline{x} = \underline{b}$. So the set of all images is just the set consisting of those vectors $\underline{b} \in \mathbb{R}^3$ for which the system $A\underline{x} = \underline{b}$ is solvable. According to part (a) the foregoing system is solvable only if $b_3 - 2b_2 + 10b_1 = 0$.

The set

$$\left\{T(\underline{x}) \mid \ \underline{x} \in \mathbb{R}^3\right\} = \left\{\underline{b} \in \mathbb{R}^3 \mid b_3 - 2b_2 + 10b_1 = 0\right\}$$

represents a plane in the set $\mathbb{R}^3$ containing the origin.

5.4 Let $\underline{y} = T(\underline{x})$. If φ is the angle between the vector $\underline{x}$ and the positive horizontal axis, and if $r = \|\underline{x}\| = \|\underline{y}\|$, then

$$\begin{cases} x_1 = r\cos\varphi \\ x_2 = r\sin\varphi, \end{cases}$$

while

$$\begin{cases} y_1 = r\cos(2\vartheta - \varphi) \\ y_2 = r\sin(2\vartheta - \varphi). \end{cases}$$

Hence,

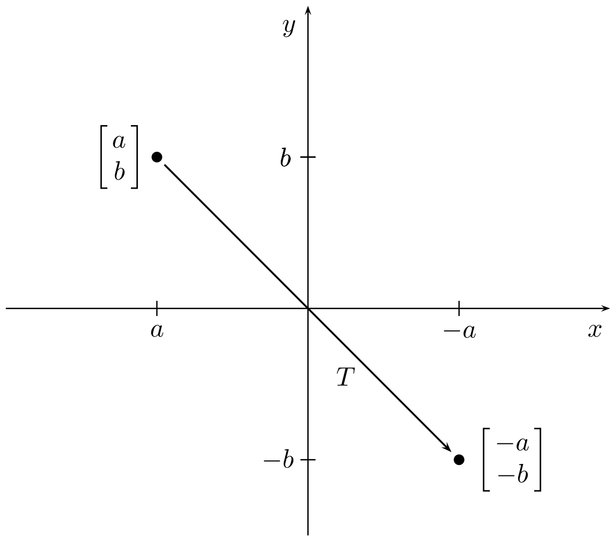
$$\begin{aligned} H_\vartheta(\underline{x}) = \underline{y} &= \begin{bmatrix} r\cos 2\vartheta\cos(-\varphi) - r\sin 2\vartheta\sin(-\varphi) \\ r\cos 2\vartheta\sin(-\varphi) + r\sin 2\vartheta\cos(-\varphi) \end{bmatrix} = \begin{bmatrix} x_1\cos 2\vartheta + x_2\sin 2\vartheta \\ -x_2\cos 2\vartheta + x_1\sin 2\vartheta \end{bmatrix} \\ &= \begin{bmatrix} \cos 2\vartheta & \sin 2\vartheta \\ \sin 2\vartheta & -\cos 2\vartheta \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}. \end{aligned}$$

So the mapping H_ϑ is in fact a left-multiplication by the given matrix¿

5.5 Since T is a linear mapping,

$$\begin{aligned} T(2\underline{v}_1 - 3\underline{v}_2 + 4\underline{v}_3) &= T(2\underline{v}_1 - 3\underline{v}_2) + T(4\underline{v}_3) = T(2\underline{v}_1) + T(-3\underline{v}_2) + T(4\underline{v}_3) \\ &= 2T(\underline{v}_1) - 3T(\underline{v}_2) + 4T(\underline{v}_3) = \begin{bmatrix} 2 \\ -2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ -9 \\ -6 \end{bmatrix} + \begin{bmatrix} -12 \\ 4 \\ 8 \end{bmatrix} = \begin{bmatrix} -10 \\ -7 \\ 6 \end{bmatrix}. \end{aligned}$$

5.7 The mapping can be presented in the following way:



(a) By means of the foregoing figure one finds that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ -y \end{bmatrix}.$$

(b) Let $\underline{u}, \underline{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$T(\underline{u}+\underline{v}) = T\left(\begin{bmatrix} u_1+v_1 \\ u_2+v_2 \end{bmatrix}\right) = \begin{bmatrix} -u_1-v_1 \\ -u_2-v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ -v_2 \end{bmatrix} = T(\underline{u}) + T(\underline{v})$$

and

$$T(c\,\underline{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} -cu_1 \\ -cu_2 \end{bmatrix} = c\begin{bmatrix} -u_1 \\ -u_2 \end{bmatrix} = cT(\underline{u}).$$

So according to the definition T is a linear mapping.

5.10 Note that, according to the definition, the ith column of A equals $T(\underline{e}_i)$.

Furthermore

$$T(\underline{e}_i) = B\underline{e}_i = [\underline{b}_1\ \underline{b}_2\ \ldots\ \underline{b}_n]\underline{e}_i = \underline{b}_i,$$

is the ith column of B.

Since for every i the ith column of B is equal to the ith column of A, it follows that $A = B$.

5.11 Let A be the matrix with the vectors $\underline{v}_1, \underline{v}_2$ and $\underline{v}_3$ as its columns:

$$A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}.$$

(a), (b) In part (a) we have to prove that the system $A\underline{x} = \underline{b}$ is solvable for any $\underline{b} \in \mathbb{R}^3$ and in part (b) we have to show that this system has a unique solution for all $\underline{b} \in \mathbb{R}^3$.

Therefore we reduce the augmented coefficient matrix of the system $A\underline{x} = \underline{b}$:

$$\begin{bmatrix} 1 & 1 & 1 & b_1 \\ 1 & 1 & 0 & b_2 \\ 1 & 0 & 0 & b_3 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 0 & -1 & b_2 - b_1 \\ 0 & -1 & -1 & b_3 - b_1 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 1 & b_1 \\ 0 & 1 & 1 & b_1 - b_3 \\ 0 & 0 & -1 & b_2 - b_1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & b_3 \\ 0 & 1 & 1 & b_1 - b_3 \\ 0 & 0 & 1 & b_1 - b_2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & b_3 \\ 0 & 1 & 0 & b_2 - b_3 \\ 0 & 0 & 1 & b_1 - b_2 \end{bmatrix}.$$

So the system $A\underline{x} = \underline{b}$ has a unique solution for any $\underline{b} \in \mathbb{R}^3$.

(c) Let $\underline{x} \in \mathbb{R}^3$. Then by the foregoing parts, we can write x as a (unique) linear combination of the vectors $\underline{v}_1, \underline{v}_2$ and $\underline{v}_3$:

$$\underline{x} = x_3\,\underline{v}_1 + (x_2 - x_3)\underline{v}_2 + (x_1 - x_2)\underline{v}_3.$$

So, by the fact that T is linear,

$$\begin{aligned} T(\underline{x}) &= T(x_3\,\underline{v}_1 + (x_2 - x_3)\underline{v}_2 + (x_1 - x_2)\underline{v}_3) \\ &= x_3 T(\underline{v}_1) + (x_2 - x_3)T(\underline{v}_2) + (x_1 - x_2)T(\underline{v}_3) \\ &= x_3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + (x_2 - x_3)\begin{bmatrix} 2 \\ -1 \end{bmatrix} + (x_1 - x_2)\begin{bmatrix} 4 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 4x_1 - 2x_2 - x_3 \\ 3x_1 - 4x_2 + x_3 \end{bmatrix}. \end{aligned}$$

5.14 (a) Note that $T(\underline{x}) = A\underline{x}$, where

$$A = \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix}.$$

In view of Theorem 3(b) we have to investigate for which values of p the trivial solution is the only solution of the system $A\underline{x} = \underline{0}$.

Reduction of the coefficient matrix of this system leads to

$$\begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 4 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & p \\ 0 & 1 & 4 \\ 0 & 0 & 1-p \end{bmatrix}.$$

The trivial solution is the only solution of the system if $p \neq 1$. For those values of p the mapping T is one-to-one.

(b) In view of Theorem 3(a) we have to investigate for which values of p the system $A\underline{x} = \underline{b}$ has a solution for every $\underline{b} \in \mathbb{R}^3$.

(Partial) reduction of the augmented coefficient matrix leads to

$$\begin{bmatrix} 1 & 0 & p & b_1 \\ 0 & 1 & 4 & b_2 \\ 1 & 0 & 1 & b_3 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & p & b_1 \\ 0 & 1 & 4 & b_2 \\ 0 & 0 & 1-p & b_3 - b_1 \end{bmatrix}.$$

If $p = 1$ the system $A\underline{x} = \underline{b}$ is solvable only if $b_1 = b_3$. So the system is solvable for all $\underline{b} \in \mathbb{R}^3$ if $p \neq 1$. For those values of p the mapping T is surjective.