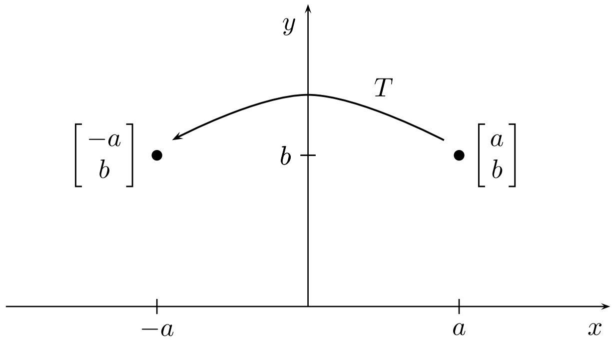
5.6 The mapping can be presented in the following way:



(a) By means of the foregoing figure one finds that

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} -x \\ y \end{bmatrix}.$$

(b) Let $\underline{u}, \underline{v} \in \mathbb{R}^2$ and $c \in \mathbb{R}$. Then

$$T(\underline{u} + \underline{v}) = T\left(\begin{bmatrix} u_1 + v_1 \\ u_2 + v_2 \end{bmatrix}\right) = \begin{bmatrix} -u_1 - v_1 \\ u_2 + v_2 \end{bmatrix} = \begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} -v_1 \\ v_2 \end{bmatrix} = T(\underline{u}) + T(\underline{v}),$$

and

$$T(c\,\underline{u}) = T\left(\begin{bmatrix} cu_1 \\ cu_2 \end{bmatrix}\right) = \begin{bmatrix} -cu_1 \\ cu_2 \end{bmatrix} = c\begin{bmatrix} -u_1 \\ u_2 \end{bmatrix} = cT(\underline{u}).$$

According to the definition T is a linear mapping.

5.8 We consider the mapping $T\colon \mathbb{R}^n \to \mathbb{R}^m$ defined by

$$T(\underline{x}) = A\underline{x} + \underline{b},$$

where A is an $m \times n$ matrix and $\underline{b} \in \mathbb{R}^m$.

We will prove that T is linear if and only if $\underline{b} = \underline{0}$.

(a) Assume that T is linear. Then on the one hand

$$T(0 \cdot \underline{u}) = 0 \cdot T(\underline{u}) = \underline{0}$$

and on the other hand

$$T(0 \cdot \underline{u}) = A\underline{0} + \underline{b} = \underline{b}.$$

So $\underline{b} = \underline{0}$.

(b) Assume that $\underline{b} = \underline{0}$. Then T is the left multiplication by a matrix. So T is linear.

5.9 (a) As observed in Chapter 3, a vector $\underline{x} \in \mathbb{R}^n$ can be written as a linear combination of the unit vectors:

$$\underline{x} = x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + \cdots + x_n\,\underline{e}_n.$$

Since the mapping T is linear, the observation preceding this exercise implies that for all $\underline{x} \in \mathbb{R}^n$

$$\begin{aligned} T(\underline{x}) &= T(x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + \cdots + x_n\,\underline{e}_n) = x_1 T(\underline{e}_1) + x_2 T(\underline{e}_2) + \cdots + x_n T(\underline{e}_n) \\ &= x_1\,\underline{e}_1 + x_2\,\underline{e}_2 + \cdots + x_n\,\underline{e}_n = \underline{x}. \end{aligned}$$

So T is the identical mapping.

(b) The mapping T is the left-multiplication by the matrix $\begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$.

5.12 (a) We give a prove based on the definition (of a linear map).

Let $\underline{u}, \underline{v} \in \mathbb{R}^n$ and $c \in \mathbb{R}$. Then

$$\begin{aligned}(T_1+T_2)(\underline{u}+\underline{v}) &= T_1(\underline{u}+\underline{v}) + T_2(\underline{u}+\underline{v}) \quad \text{(according to the definition of } T_1{+}T_2\text{)} \\ &= T_1(\underline{u}) + T_1(\underline{v}) + T_2(\underline{u}) + T_2(\underline{v}) \quad \text{(because } T_1 \text{ and } T_2 \text{ are linear)} \\ &= T_1(\underline{u}) + T_2(\underline{u}) + T_1(\underline{v}) + T_2(\underline{v}) \\ &= (T_1+T_2)(\underline{u}) + (T_1+T_2)(\underline{v}),\end{aligned}$$

and

$$(T_1+T_2)(c\,\underline{u}) = T_1(c\,\underline{u}) + T_2(c\,\underline{u}) = c\,T_1(\underline{u}) + c\,T_2(\underline{u}) = c\,[T_1(\underline{u}) + T_2(\underline{u})] = c\,(T_1+T_2)(\underline{u}).$$

Next we will give a proof by using the standard matrix.

Let A_1 be the standard matrix corresponding to T_1 and let A_2 be the standard matrix corresponding to T_2. Then $T_1(\underline{u}) = A_1\underline{u}$ and $T_2(\underline{u}) = A_2\underline{u}$ for all $\underline{u} \in \mathbb{R}^n$. This however means that for every $\underline{u} \in \mathbb{R}^n$

$$(T_1+T_2)(\underline{u}) = T_1(\underline{u}) + T_2(\underline{u}) = A_1\underline{u} + A_2\underline{u} = (A_1+A_2)\underline{u}.$$

This proves that the mapping T_1+T_2 is the left multiplication by the matrix A_1+A_2. Hence the mapping T_1+T_2 is linear.

(b) For a vector $\underline{v} \in \text{Ker}(T_1) \cap \text{Ker}(T_2)$ it holds that $T_1(\underline{v}) = T_2(\underline{v}) = \underline{0}$. By consequence,

$$(T_1+T_2)(\underline{v}) = T_1(\underline{v}) + T_2(\underline{v}) = \underline{0} + \underline{0} = \underline{0},$$

in other words: $\underline{v} \in \text{Ker}(T_1+T_2)$.

(c) Consider the mappings $T_1\colon \mathbb{R}^2 \to \mathbb{R}^2$ and $T_2\colon \mathbb{R}^2 \to \mathbb{R}^2$ defined by

$$T_1(\underline{x}) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\underline{x} \qquad \text{and} \qquad T_2(\underline{x}) = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}\underline{x} \qquad (\underline{x} \in \mathbb{R}^2).$$

Then $\text{Ker}(T_1) = \text{Ker}(T_2) = \{\underline{0}\}$. However, for every $\underline{x} \in \mathbb{R}^2$ it holds that

$$(T_1+T_2)\underline{x} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}\underline{x} = \underline{0},$$

which implies that $\text{Ker}(T_1+T_2) = \mathbb{R}^2$.

5.13 (a) Let $\underline{u}, \underline{v} \in \mathbb{R}^2$. Then

$$T(\underline{u}) = T(\underline{v}) \Longleftrightarrow \begin{bmatrix} u_1 \\ u_2 \\ u_1+u_2 \end{bmatrix} = \begin{bmatrix} v_1 \\ v_2 \\ v_1+v_2 \end{bmatrix} \Longleftrightarrow \begin{cases} u_1 = v_1 \\ u_2 = v_2 \end{cases} \Longleftrightarrow \underline{u} = \underline{v}.$$

Hence, the mapping T is one-to-one.

(b) Since

$$T(\underline{0}) = T\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right),$$

the mapping T is not one-to-one.

5.15 (a) Let $x, y \in \mathbb{R}$. Then

$$T\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence, the mapping T is the left-multiplication by the matrix

$$A = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}.$$

So the mapping T is linear.

(b) Note that

$$\mathrm{Im}(T) = \mathrm{Col}(A) = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}\right\} = \mathrm{span}\left\{\begin{bmatrix} 1 \\ 1 \end{bmatrix}\right\}.$$

This set corresponds with the line $y = x$ in the x, y-plane.

Note that $\mathrm{Ker}(T) = \mathrm{Null}(A)$. Since

$$A\underline{x} = \underline{0} \Longleftrightarrow x_1 = 0 \Longleftrightarrow \underline{x} = c\,\underline{e}_2 \text{ for some } c \in \mathbb{R},$$

$\mathrm{Ker}(T) = \mathrm{Null}(A) = \{c\,\underline{e}_2 |\ c \in \mathbb{R}\} = \mathrm{span}\{\underline{e}_2\}$.

This set corresponds with the line $x = 0$ in the x, y-plane.