6.5 According to Theorem 4 (d),

$$(A\underline{u}) \cdot (B\underline{v}) = (A\underline{u})^T B\underline{v} = \underline{u}^T A^T B\underline{v} = \underline{u} \cdot (A^T B\underline{v}).$$

6.8 (a) On the one hand it holds that $(AB)^2 = ABAB$, while on the other hand $A^2B^2 = AABB$. Because, in general, $BA \neq AB$, we try to find a counterexample.

If $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix}$, then

$$AB = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad (AB)^2 = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix},$$

while

$$A^2 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \quad B^2 = \begin{bmatrix} 1 & 0 \\ 1 & 0 \end{bmatrix} \qquad \text{and} \qquad A^2B^2 = \begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix}.$$

(b) Note that

$$(A+B)^2 = (A+B)(A+B) = A(A+B) + B(A+B) = A^2 + AB + BA + B^2.$$

So $(A+B)^2 = A^2 + 2AB + B^2$ if and only if $AB + BA = 2AB \iff BA = AB$.

Hence, by using the same matrices as in the foregoing part one can show that the statement is not true.

6.9 (a) Note that

$$a_{11} = a_{12} = a_{21} = a_{22} = a_{23} = a_{32} = a_{33} = a_{34} = a_{43} = -1$$

and

$$a_{13} = a_{31} = a_{14} = a_{41} = a_{24} = a_{42} = 1.$$

So

$$A = \begin{bmatrix} -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \\ 1 & -1 & -1 & -1 \\ 1 & 1 & -1 & -1 \end{bmatrix} \quad \text{and} \quad A^2 = \begin{bmatrix} 4 & 2 & -2 & -4 \\ 2 & 4 & 0 & -2 \\ -2 & 0 & 4 & 2 \\ -4 & -2 & 2 & 4 \end{bmatrix}.$$

(b) Obviously, $\text{tr}(A) = -4$.

6.10 For $n \times n$ matrices C and D it holds that

$$\text{tr}(CD) = \sum_{i=1}^{n}(CD)_{ii} = \sum_{i=1}^{n}\Big(\sum_{j=1}^{n} c_{ij}d_{ji}\Big) = \sum_{i=1}^{n}\sum_{j=1}^{n} c_{ij}d_{ji} = \sum_{j=1}^{n}\Big(\sum_{i=1}^{n} d_{ji}c_{ij}\Big) = \sum_{j=1}^{n}(DC)_{jj} = \text{tr}(DC).$$

6.12 (a) As

$$M^T = \begin{bmatrix} A & O \\ O & B \end{bmatrix}^T = \begin{bmatrix} A^T & O \\ O & B^T \end{bmatrix},$$

the matrix M is symmetric if and only if $A = A^T$ and $B = B^T$, that is: if and only if the matrices A and B are symmetric.

(b) We compare the entries at the position (i, j) of both matrices.

Case 1: $i = 1, 2$ and $j = 1, 2$.

The entry at the position (i, j) of the matrix

$$MN = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \begin{bmatrix} C & O \\ O & D \end{bmatrix}$$

equals

$$m_{i1}n_{1j} + \cdots + m_{i4}n_{4j} = \left(A\right)_{i1}\left(C\right)_{1j} + \left(A\right)_{i2}\left(C\right)_{2j} + 0 \cdot 0 + 0 \cdot 0 = \left(AC\right)_{ij}.$$

This is the entry at the position (i, j) of the matrix

$$\begin{bmatrix} AC & O \\ O & BD \end{bmatrix}.$$

Case 2: $i = 1, 2$ and $j = 3, 4$.

The entry at the position (i, j) of the matrix

$$MN = \begin{bmatrix} A & O \\ O & B \end{bmatrix} \begin{bmatrix} C & O \\ O & D \end{bmatrix}$$

equals

$$m_{i1}n_{1j} + \cdots + m_{i4}n_{4j} = \left(A\right)_{i1} \cdot 0 + \left(A\right)_{i2} \cdot 0 + 0 \cdot \left(D\right)_{1,j-2} + 0 \cdot \left(D\right)_{2,j-2} = 0.$$

This is the entry at the position (i, j) of the matrix

$$\begin{bmatrix} AC & O \\ O & BD \end{bmatrix}.$$

The other cases are similar.

6.13 (a) First note that the matrices AA^T and $A + A^T$ are well-defined.

Further, according to Theorem 4,

$$(AA^T)^T \overset{\text{part (d)}}{=} (A^T)^T A^T \overset{\text{part (a)}}{=} AA^T$$

and

$$(A + A^T)^T \overset{\text{part(b)}}{=} A^T + (A^T)^T \overset{\text{part (a)}}{=} A^T + A = A + A^T.$$

(b) If the matrix A is symmetric, then

$$(A^2)^T = (AA)^T = A^T A^T = AA = A^2.$$

This means that the matrix A^2 is symmetric.