7.1 (a) It is easy to check that

$$\begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 2 & 3 \\ 3 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence $A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix}$.

(b) Note that

$$A^{-2} = A^{-1}A^{-1} = \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 5 & -3 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} 34 & -21 \\ -21 & 13 \end{bmatrix}.$$

(c) Since the inverse of a matrix is unique, the equality $(I + 2B)^{-1} = A^{-1}$ implies that $I + 2B = A$.

Hence,

$$B = \tfrac{1}{2}(A - I) = \tfrac{1}{2} \begin{bmatrix} 1 & 3 \\ 3 & 4 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & \frac{3}{2} \\ \frac{3}{2} & 2 \end{bmatrix}.$$

7.3 The inequality $d_1 d_2 \cdots d_n \neq 0$ implies that $d_i \neq 0$ for every i.

So the matrix

$$B = \begin{bmatrix} \frac{1}{d_1} & 0 & \cdots & 0 \\ 0 & \frac{1}{d_2} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \frac{1}{d_n} \end{bmatrix}$$

is well defined. Since $BD = DB = I$, the matrix B is the inverse of the matrix D: $B = D^{-1}$.

7.4 Because $A(3I - A) = 3A - A^2 = I$ and $(3I - A)A = 3A - A^2 = I$, the matrix A is invertible and $A^{-1} = 3I - A$.

7.5 For $n \in \mathbb{N}$ we introduce the statement $\mathcal{P}(n)$: the matrix A^n is invertible and $(A^n)^{-1} = (A^{-1})^n$.

Since $A^{-1} = A^{-1}$, the statement is true for $n = 1$.

Let $k \in \mathbb{N}$ and suppose that the statement $\mathcal{P}(k)$ is true.

So the matrix A^k is invertible and $(A^k)^{-1} = (A^{-1})^k$.

Then, according to Theorem 2 (a), the matrix $A^{k+1} = A^k \cdot A$ is invertible and

$$\left(A^{k+1}\right)^{-1} = (A^k \cdot A)^{-1} = A^{-1} \cdot \left(A^k\right)^{-1} = A^{-1} \cdot \left(A^{-1}\right)^k = \left(A^{-1}\right)^{k+1}.$$

Hence the statement $\mathcal{P}(k+1)$ is true.

So, by the Principle of Induction, the statement $\mathcal{P}(n)$ is true for all $n \in \mathbb{N}$.

7.6 The matrix B arises from the matrix A by subtracting two times the first row from the third one.

If we apply this operation to the 3×3 identity matrix I, then the result is the elementary matrix

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix}.$$

Note that $EA = B$.

By using the inverse operation we obtain the matrix

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

Note that $E^{-1}B = A$.

7.8 Because the matrix AB is invertible, the matrix $(AB)^{-1}$ is well-defined, while

$$AB(AB)^{-1} = I \quad \text{and} \quad (AB)^{-1}AB = I.$$

Since the matrix A is square and $A[B(AB)^{-1}] = I$, according to Theorem 7, the matrix A is invertible and $A^{-1} = B(AB)^{-1}$.

Since the matrix B is square, and $[(AB)^{-1}A] \cdot B = I$, according to Theorem 7, the matrix B is invertible and $B^{-1} = (AB)^{-1}A$.

7.10 (a) (Partial) reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} -1 & 3 & -4 & 1 & 0 & 0 \\ 2 & 4 & 1 & 0 & 1 & 0 \\ -4 & 2 & -9 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & -10 & 7 & -4 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -3 & 4 & -1 & 0 & 0 \\ 0 & 10 & -7 & 2 & 1 & 0 \\ 0 & 0 & 0 & -2 & 1 & 1 \end{bmatrix}.$$

In view of the third row of this matrix, the matrix A is not invertible.

(b) (Partial) reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 3 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 3 & 5 & 0 & 0 & 0 & 1 & 0 \\ 1 & 3 & 5 & 7 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 0 & -1 & 0 & 1 & 0 \\ 0 & 3 & 5 & 7 & -1 & 0 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 5 & 7 & 0 & -1 & 0 & 1 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 & -1 & 1 & 0 & 0 \\ 0 & 0 & 5 & 0 & 0 & -1 & 1 & 0 \\ 0 & 0 & 0 & 7 & 0 & 0 & -1 & 1 \end{bmatrix}.$$

So the inverse of the matrix A exists and

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ -\frac{1}{3} & \frac{1}{3} & 0 & 0 \\ 0 & -\frac{1}{5} & \frac{1}{5} & 0 \\ 0 & 0 & -\frac{1}{7} & \frac{1}{7} \end{bmatrix}.$$

(c) (Partial) reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} k_4 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & k_3 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & k_2 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & k_1 & 1 & 0 & 0 & 0 \end{bmatrix}.$$

So the inverse of the matrix A exists if $k_i \neq 0$ for all i and in that case

$$A^{-1} = \begin{bmatrix} 0 & 0 & 0 & \frac{1}{k_4} \\ 0 & 0 & \frac{1}{k_3} & 0 \\ 0 & \frac{1}{k_2} & 0 & 0 \\ \frac{1}{k_1} & 0 & 0 & 0 \end{bmatrix}.$$

7.12 Because $T\colon \mathbb{R}^n \to \mathbb{R}^n$ is a linear mapping, the standard matrix A corresponding to T is a (square) $n \times n$ matrix.

The following holds:

$$\begin{aligned} \text{the map } T \text{ is surjective} &\overset{\text{by definition}}{\iff} \text{there exists for all } \underline{b} \in \mathbb{R}^n \text{ an } \underline{x} \in \mathbb{R}^n \text{ such that } T(\underline{x}) = \underline{b} \\ &\iff \text{there exists for all } \underline{b} \in \mathbb{R}^n \text{ an } \underline{x} \in \mathbb{R}^n \text{ such that } A\underline{x} = \underline{b} \\ &\iff \text{the system } A\underline{x} = \underline{b} \text{ is solvable for every } \underline{b} \in \mathbb{R}^n \\ &\overset{\text{Theorem 8}}{\iff} A \text{ is invertible.} \end{aligned}$$

7.13 According to the Theorems 9 and 10 the mapping T is surjective and one-to-one.

Therefore, for every $\underline{w} \in \mathbb{R}^n$ there exists precisely one vector $\underline{v} \in \mathbb{R}^n$ such that $T(\underline{v}) = \underline{w}$; furthermore $T^{-1}(\underline{w}) = \underline{v}$.

By using the invertibility of A it follows that

$$T(\underline{v}) = \underline{w} \Longrightarrow A\underline{v} = \underline{w} \Longrightarrow \underline{v} = A^{-1}\underline{w}.$$

This means that $T^{-1}(\underline{w}) = A^{-1}\underline{w}$ for all $\underline{w} \in \mathbb{R}^n$.

Note that one cannot use the fact that T^{-1} is a left-multiplication by some matrix. First you have to prove that T^{-1} is a linear mapping.

7.14 (a) Reduction of the matrix A leads to

$$\begin{bmatrix} 5 & 2 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Hence, the matrix A is invertible. Since T is the left-multiplication by an invertible matrix, T has an inverse.

(b) According to Exercise 13, T^{-1} is the left-multiplication by the matrix A^{-1}. Reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} 5 & 2 & 1 & 0 \\ 2 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 2 & 1 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 & -2 \\ 0 & 1 & -2 & 5 \end{bmatrix}.$$

So, for $\underline{x} \in \mathbb{R}^2$,

$$T^{-1}(\underline{x}) = A^{-1}(\underline{x}) = \begin{bmatrix} 1 & -2 \\ -2 & 5 \end{bmatrix} \underline{x} = \begin{bmatrix} x_1 - 2x_2 \\ -2x_1 + 5x_2 \end{bmatrix}.$$