7.2 (a) Assume that A, B and C invertible.

Then, according to Theorem 2 (a) the matrix AB is invertible too and $(AB)^{-1} = B^{-1}A^{-1}$.

Because AB and C are invertible $(n \times n)$ matrices, again according to Theorem 2 (a), the matrix $(AB)C$ is invertible and

$$((AB)C)^{-1} = C^{-1}(AB)^{-1}.$$

Hence, ABC is invertible and $(ABC)^{-1} = C^{-1}(AB)^{-1} = C^{-1}B^{-1}A^{-1}$.

(b) If A is invertible, then

$$AB = O \Longrightarrow A^{-1}(AB) = A^{-1}O \Longrightarrow (A^{-1}A)B = O \Longrightarrow B = O.$$

(c) Assume that B and AB are invertible.

Because AB is invertible, the matrix $(AB)^{-1}$ is well-defined, while furthermore $AB(AB)^{-1} = I$ and $(AB)^{-1}AB = I$.

Because the matrix B is invertible, the second equation implies that

$$\begin{aligned}(AB)^{-1}AB = I &\Longrightarrow [(AB)^{-1}AB] \cdot B^{-1} = I \cdot B^{-1} \\ &\Longrightarrow (AB)^{-1}A = B^{-1} \\ &\Longrightarrow B \cdot (AB)^{-1}A = B \cdot B^{-1} \\ &\Longrightarrow [B(AB)^{-1}] \cdot A = I,\end{aligned}$$

while the first equation can be written as $A \cdot [B(AB)^{-1}] = I$.

This means that the matrix A is invertible and that $A^{-1} = B(AB)^{-1}$.

(d) If A is invertible, then according to Theorem 1 (b), the matrix A^T is invertible. Hence, in view of Theorem 2 (a), the matrix AA^T is invertible.

(e) The statement is not true. Let $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$, then $AB = O$ and $A \neq O$ and $B \neq O$.

7.7 Assume that $AB = I$. Then the proof of Theorem 7 implies that B is invertible and that $B^{-1} = A$. Theorem 1 (a) implies that $A = B^{-1}$ is invertible and that $A^{-1} = \left(B^{-1}\right)^{-1} = B$.

7.9 Let M' be the 4×4 matrix defined by

$$M' = \begin{bmatrix} A^{-1} & O \\ O & B^{-1} \end{bmatrix}.$$

According to Exercise 6.12,

$$MM' = \begin{bmatrix} AA^{-1} & O \\ O & BB^{-1} \end{bmatrix} = \begin{bmatrix} I & O \\ O & I \end{bmatrix} = I.$$

In view of Theorem 7, this implies that the matrix M is invertible and that $M^{-1} = M'$.

7.10 (d) (Partial) reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} 1 & 0 & 5 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 0 \\ 3 & 2 & 6 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 2 & -9 & -3 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 5 & 1 & 0 & 0 \\ 0 & 1 & -5 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}$$
$$\to \begin{bmatrix} 1 & 0 & 0 & 6 & 10 & -5 \\ 0 & 1 & 0 & -6 & -9 & 5 \\ 0 & 0 & 1 & -1 & -2 & 1 \end{bmatrix}.$$

So the inverse of the matrix A exists and

$$A^{-1} = \begin{bmatrix} 6 & 10 & -5 \\ -6 & -9 & 5 \\ -1 & -2 & 1 \end{bmatrix}.$$

(e) (Partial) reduction of the matrix $[\,A \quad I\,]$ leads to

$$\begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ -1 & 5 & 6 & 0 & 1 & 0 \\ 5 & -4 & 5 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 6 & 10 & -5 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & -2 & -1 & 1 & 0 & 0 \\ 0 & 3 & 5 & 1 & 1 & 0 \\ 0 & 0 & 0 & -7 & -2 & 1 \end{bmatrix}.$$

According to the last row of this matrix, the matrix A is not invertible.

7.11 (a) Assume that the system $A\underline{x} = \underline{0}$ only has the trivial solution. Then, according to Theorem 6, the matrix A is invertible. So, by Theorem 2 (b), the matrix A^k is invertible too. By using Theorem 6 again, this implies that the system $A^k\underline{x} = \underline{0}$ only has the trivial solution.

(b) Assume that the system $A\underline{x} = \underline{0}$ only has the trivial solution. Then, according to Theorem 6, the matrix A is invertible. So, by Theorem 2 (b), the matrix QA is invertible too. By using Theorem 6 again, this implies that the system $QA\underline{x} = \underline{0}$ only has the trivial solution.

Assume that the system $QA\underline{x} = \underline{0}$ only has the trivial solution.

We will show that the system $A\underline{x} = \underline{0}$ only has the trivial solution too.

So suppose that $A\underline{s} = \underline{0}$. Then $QA\underline{s} = Q\underline{0} = \underline{0}$. Hence, $\underline{s}$ is a solution of the system $QA\underline{x} = \underline{0}$. Then however, $\underline{s} = \underline{0}$.

7.15 (a) Note that

$$T(\underline{x}) = \begin{bmatrix} a_1x_1 \\ a_2x_2 \\ \vdots \\ a_nx_n \end{bmatrix} = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}.$$

So T is the left-multiplication by the matrix

$$A = \begin{bmatrix} a_1 & 0 & \cdots & 0 \\ 0 & a_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n \end{bmatrix}.$$

According to the Theorems 9 and 6, the mapping T is one-to-one if and only if the system $A\underline{x} = \underline{0}$ only has the trivial solution.

Obviously

$$A\underline{x} = \underline{0} \Longleftrightarrow \begin{bmatrix} a_1 x_1 \\ a_2 x_2 \\ \vdots \\ a_n x_n \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

only has the trivial solution if $a_i \neq 0$ for all i.

(b) According to Theorem 10 and the foregoing part, the mapping is surjective if $a_i \neq 0$ for all i.

(c) If $a_i \neq 0$ for all i, then

$$A^{-1} = \begin{bmatrix} a_1^{-1} & 0 & \cdots & 0 \\ 0 & a_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_n^{-1} \end{bmatrix}.$$

So, for $\underline{x} \in \mathbb{R}^n$,

$$T^{-1}(\underline{x}) = A^{-1}\underline{x} = \begin{bmatrix} x_1/a_1 \\ x_2/a_2 \\ \vdots \\ x_n/a_n \end{bmatrix}.$$