8.1 Since $\det A = (a-3)(a-2) - 20 = a^2 - 5a - 14 = (a-7)(a+2)$, the determinant of the matrix A is equal to zero if $a = -2$ or $a = 7$.

8.2 Expansion along the first row leads to

(a) $1 \cdot \det \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} = 1 - 0 \cdot k = 1.$

(b) $k \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = k.$

(c) $-1 \cdot \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = -1.$

8.3 (a) Expansion along the second column leads to

$$\begin{aligned} -(-2) \cdot \det \begin{bmatrix} 6 & -1 \\ -3 & 4 \end{bmatrix} + 7 \cdot \det \begin{bmatrix} 1 & 3 \\ -3 & 4 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix} &= 2(24-3) + 7(4+9) - (-1-18) \\ &= 42 + 91 + 19 = 152. \end{aligned}$$

(b) Expansion along the third row leads to

$$\begin{aligned} (-3) \cdot \det \begin{bmatrix} -2 & 3 \\ 7 & -1 \end{bmatrix} - 1 \cdot \det \begin{bmatrix} 1 & 3 \\ 6 & -1 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 1 & -2 \\ 67 & \end{bmatrix} &= -3(2-21) - (-1-18) + 4(7+12) \\ &= 57 + 19 + 76 = 152. \end{aligned}$$

(c) Expansion along the third column leads to

$$\begin{aligned} 3 \cdot \det \begin{bmatrix} 6 & 7 \\ -3 & 1 \end{bmatrix} - (-1) \cdot \det \begin{bmatrix} 1 & -2 \\ -3 & 1 \end{bmatrix} + 4 \cdot \det \begin{bmatrix} 1 & -2 \\ 6 & 7 \end{bmatrix} &= 3(6+21) + (1-6) + 4(7+12) \\ &= 81 - 5 + 76 = 152. \end{aligned}$$

8.4 By expanding along the first row each time we obtain

$$4 \cdot \det \begin{bmatrix} -1 & 0 & 0 \\ 6 & 3 & 0 \\ -8 & 4 & -3 \end{bmatrix} = 4 \cdot -1 \cdot \det \begin{bmatrix} 3 & 0 \\ 4 & -3 \end{bmatrix} = 4 \cdot -1 \cdot -9 = 36.$$

8.5 We introduce for $n \in \mathbb{N}$ the statement $\mathcal{P}(n)$: for an $n \times n$ matrix A having the triangular form $\det A = a_{11} a_{22} \cdots a_{nn}$.

(1) First we show that the statement $P(1)$ is true: if $A = [a_{11}]$, then $\det A = a_{11}$.

(2) Let $k \in \mathbb{N}$ and suppose that the statement $P(k)$ is true; so if an $k \times k$ matrix A has the triangular form, then $\det A = a_{11} a_{22} \cdots a_{kk}$.

Let A' be an $(k+1) \times (k+1)$ matrix having the triangular form.

By expanding along the first column (row) if A has an upper triangular form (lower triangular form), we obtain

$$\det A' = a'_{11} \cdot \det A'_{11} = a'_{11} \cdot [a'_{22} a'_{33} \cdots a'_{(k+1)(k+1)}] = a'_{11} \cdot a'_{22} \cdots a'_{(k+1)(k+1)}.$$

This proves that the statement $P(k+1)$ is true.

According to the Principle of Induction, the statement $P(n)$ is true for every $n \in \mathbb{N}$.

8.6 (a) Matrix A' is obtained from matrix A by interchanging the two rows. So $A' = E_1 A$, where

$$E_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Furthermore, $\det A = ad - bc$ and $\det A' = bc - ad = -\det A$.

(b) Matrix A' is obtained from matrix A by adding the k-multiple of the first row to the second one. So $A' = E_2 A$, where

$$E_2 = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}.$$

Furthermore, $\det A = 18 - 20 = -2$ and $\det A' = 18 + 12k - (20 + 12k) = -2 = \det A$.

(c) Matrix A' is obtained from matrix A by multiplying row 1 by k. So $A' = E_3 A$, where

$$E_3 = \begin{bmatrix} k & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Furthermore, $\det A = 1 \det A_{11} - 1 \det A_{12} + 1 \det A_{13} = -5$ and $\det A' = k \det A_{11} - k \det A_{12} + k \det A_{13} = k \det A = -5k$.

8.8 (a) Reduction leads to

$$\det \begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 2 \\ 3 & 2 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 3 & 2 & 4 \end{bmatrix} = -\det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 3 & 1 \\ 0 & -1 & -2 \end{bmatrix} = -3 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & -1 & -2 \end{bmatrix}$$
$$= -3 \cdot \det \begin{bmatrix} 1 & 1 & 2 \\ 0 & 1 & \frac{1}{3} \\ 0 & 0 & -\frac{5}{3} \end{bmatrix} = -3 \cdot -\tfrac{5}{3} = 5.$$

(b) Reduction leads to

$$\det \begin{bmatrix} 2 & 1 & 3 & 1 \\ 1 & 0 & 1 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 2 & 1 & 3 & 1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 2 & 1 & 0 \\ 0 & 1 & 2 & 3 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & -1 & 2 \\ 0 & 0 & 1 & 4 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 0 & 0 & 1 & -2 \\ 0 & 0 & 0 & 6 \end{bmatrix} = 6.$$

8.11 (a) Since

$$\det \begin{bmatrix} 1 & 0 & -1 \\ 9 & -1 & 4 \\ 8 & 9 & -1 \end{bmatrix} = \det \begin{bmatrix} 1 & 0 & -1 \\ 0 & -1 & 13 \\ 0 & 9 & 7 \end{bmatrix} = -\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -13 \\ 0 & 9 & 7 \end{bmatrix}$$
$$= -\det \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & -13 \\ 0 & 0 & 124 \end{bmatrix} = -124 \neq 0,$$

the given matrix is invertible.

(b) By expanding along the second column of the matrix, it appears that the determinant of the matrix is equal to zero. Hence the given matrix is not invertible.

8.12 (a) Note that

$$\det A = 0 \Longleftrightarrow a^2 - 5a + 6 - 4 = 0 \Longleftrightarrow a^2 - 5a + 2 = 0 \Longleftrightarrow a = \frac{5 \pm \sqrt{25-8}}{2} = 2\tfrac{1}{2} \pm \tfrac{1}{2}\sqrt{17}.$$

So the matrix A is not invertible if $a = 2\frac{1}{2} \pm \frac{1}{2}\sqrt{17}$.

8.13 Obviously,

$$\begin{aligned}\det A &= \det \begin{bmatrix} a+e & b+f \\ c & d \end{bmatrix} = (a+e)d - c(b+f) = ad - bc + ed - cf \\ &= \det \begin{bmatrix} a & b \\ c & d \end{bmatrix} + \det \begin{bmatrix} e & f \\ c & d \end{bmatrix} = \det B + \det C.\end{aligned}$$

8.15 Note that

$$\begin{aligned}\det \begin{bmatrix} a_1+b_1 & a_1-b_1 & c_1 \\ a_2+b_2 & a_2-b_2 & c_2 \\ a_3+b_3 & a_3-b_3 & c_3 \end{bmatrix} &= \det \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1-b_1 & a_2-b_2 & a_3-b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= \det \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ 2a_1 & 2a_2 & 2a_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= 2\det \begin{bmatrix} a_1+b_1 & a_2+b_2 & a_3+b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= 2\det \begin{bmatrix} b_1 & b_2 & b_3 \\ a_1 & a_2 & a_3 \\ c_1 & c_2 & c_3 \end{bmatrix} \\ &= -2\det \begin{bmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{bmatrix} = -2\det \begin{bmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{bmatrix}.\end{aligned}$$

8.16 One easily finds that

(a) $\det AB = -1 \cdot 2 = -2$.

(b) $\det B^5 = 2^5 = 32$.

(c) $\det 2A = 2^4 \cdot (-1) = -16$.

(d) $\det A^T A = (-1) \cdot (-1) = 1$.

(e) $\det B^{-1}AB = \frac{1}{2} \cdot -1 \cdot 2 = -1$.

8.18 First we evaluate the determinant of the coefficient matrix of the system:

$$\det\begin{bmatrix}1 & -3 & 1\\ 2 & -1 & 0\\ 4 & 0 & -3\end{bmatrix} = 1\cdot\det\begin{bmatrix}2 & -1\\ 4 & 0\end{bmatrix} + (-3)\cdot\det\begin{bmatrix}1 & -3\\ 2 & -1\end{bmatrix} = 4 - 15 = -11.$$

Because the determinant is not equal to zero, the system of equations has a unique solution, say $\underline{s}$. According to Cramer's rule

$$s_1 = \frac{\det\begin{bmatrix}4 & -3 & 1\\ -2 & -1 & 0\\ 0 & 0 & -3\end{bmatrix}}{-11} = \frac{-3\cdot(-10)}{-11} = -\tfrac{30}{11},$$

$$s_2 = \frac{\det\begin{bmatrix}1 & 4 & 1\\ 2 & -2 & 0\\ 4 & 0 & -3\end{bmatrix}}{-11} = \frac{1\cdot 8 + (-3)\cdot(-10)}{-11} = -\tfrac{38}{11}$$

and

$$s_3 = \frac{\det\begin{bmatrix}1 & -3 & 4\\ 2 & -1 & -2\\ 4 & 0 & 0\end{bmatrix}}{-11} = \frac{4\cdot 10}{-11} = -\tfrac{40}{11}.$$

8.20 Note that (expand along the first row)

$$\det A = (28+1) - (-2)(24-3) + 3(6+21) = 29 + 2\cdot 21 + 3\cdot 27 = 152.$$

According to Theorem 9, the entry at the position $(1,2)$ is

$$(-1)^{1+2}\frac{\det A_{21}}{\det A} = -\frac{-8-3}{152} = \tfrac{11}{152}.$$