8.7 We expand the determinant of the matrix M along the first column:

$$\det\begin{bmatrix} a_{11} & a_{12} & 0 & 0 \\ a_{21} & a_{22} & 0 & 0 \\ 0 & 0 & b_{11} & b_{12} \\ 0 & 0 & b_{21} & b_{22} \end{bmatrix} = a_{11}\cdot\det\begin{bmatrix} a_{22} & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{bmatrix} - a_{21}\cdot\det\begin{bmatrix} a_{12} & 0 & 0 \\ 0 & b_{11} & b_{12} \\ 0 & b_{21} & b_{22} \end{bmatrix}$$

$$= a_{11}a_{22}\cdot\det B - a_{21}a_{12}\cdot\det B = [a_{11}a_{22} - a_{21}a_{12}]\cdot\det B$$

$$= \det A\cdot\det B.$$

8.8 (c) Reduction leads to

$$\det\begin{bmatrix} 0 & 1 & 1 & 1 \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix} = -\det\begin{bmatrix} \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ 0 & 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix} = -\frac{1}{2}\det\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ \frac{2}{3} & \frac{1}{3} & \frac{1}{3} & 0 \\ -\frac{1}{3} & \frac{2}{3} & 0 & 0 \end{bmatrix}$$

$$= -\frac{1}{2}\det\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & -\frac{1}{3} & -1 & -\frac{2}{3} \\ 0 & 1 & \frac{2}{3} & \frac{1}{3} \end{bmatrix} = -\frac{1}{2}\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & -\frac{2}{3} & -\frac{1}{3} \\ 0 & 0 & -\frac{1}{3} & -\frac{2}{3} \end{bmatrix}$$

$$= \frac{1}{3}\det\begin{bmatrix} 1 & 1 & 2 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & \frac{1}{2} \\ 0 & 0 & 0 & -\frac{1}{2} \end{bmatrix} = -\frac{1}{6}.$$

8.9 Let $n \in \mathbb{N}$. We introduce for $k \in \mathbb{N}$ the statement $\mathcal{P}(k)$:

for an $n \times n$ matrix A and elementary matrices $E_1, E_2, \ldots, E_k$ of the same size,

$$\det(E_kE_{k-1}\cdots E_1A) = (\det E_k)(\det E_{k-1})\cdots(\det E_1)(\det A).$$

(1) First we show that the statement $P(1)$ is true: according to the Theorems 3 and 4 $\det(E_1A) = (\det E_1)(\det A)$.

(2) Let $l \in \mathbb{N}$ and suppose that the statement $P(l)$ is true; so if A is an $n\times n$ matrix and $E_1, E_2, \ldots, E_l$ are elementary matrices of the same size, then

$$\det(E_lE_{l-1}\cdots E_1A) = (\det E_l)(\det E_{l-1})\cdots(\det E_1)(\det A).$$

Let E_{l+1} be an $n \times n$ elementary matrix.

Then, by the Theorems 3 and 4 and the induction hypothesis,

$$\det(E_{l+1}E_l\cdots E_1A) = \det E_{l+1}\det(E_l\cdots E_1A)$$

$$= (\det E_{l+1})(\det E_l)(\det E_{l-1})\cdots(\det E_1)(\det A).$$

This proves that the statement $P(l+1)$ is true.

According to the Principle of Induction, the statement $P(k)$ is true for every $k \in \mathbb{N}$.

8.10 By applying Theorem 4, we find that

(a)

$$\det\begin{bmatrix} d & e & f \\ g & h & i \\ a & b & c \end{bmatrix} = -\det\begin{bmatrix} a & b & c \\ g & h & i \\ d & e & f \end{bmatrix} = \det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = -6.$$

(b)

$$\det\begin{bmatrix} -3a & -3b & -3c \\ d & e & f \\ g-4d & h-4e & i-4f \end{bmatrix} = -3\det\begin{bmatrix} a & b & c \\ d & e & f \\ g-4d & h-4e & i-4f \end{bmatrix} = -3\det\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = 18.$$

8.12 (b) Note that

$$\det A = \det\begin{bmatrix} 1 & 2 & 4 \\ 0 & -5 & -6 \\ 0 & 3-2a & 2-4a \end{bmatrix} = -5\det\begin{bmatrix} 1 & 2 & 4 \\ 0 & 1 & \frac{6}{5} \\ 0 & 3-2a & 2-4a \end{bmatrix}$$

$$= -5[1\cdot(2-4a) - \tfrac{6}{5}\cdot(3-2a)] = 8+8a.$$

So $\det A = 0 \iff 8+8a = 0 \iff a = -1$.

Hence, the matrix A is not invertible if $a=-1$.

8.14 According to Theorem 7, $\det A^{-1}\det A = \det(A^{-1}A) = \det I = 1$. Because the matrix A is invertible, $\det A \neq 0$. So

$$\det A^{-1} = \frac{1}{\det A}.$$

8.17 (a) We start wit considering the 2×2 case. Then

$$B = \begin{bmatrix} a_{11} & -a_{12} \\ -a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

So $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$.

For the general situation, let D be the $n\times n$ diagonal matrix with

$$d_{ii} = \begin{cases} 1 & \text{if } i \text{ odd} \\ -1 & \text{if } i \text{ even} \end{cases}.$$

Then

$$DAD = \begin{bmatrix} 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \end{bmatrix}\begin{bmatrix} a_{11} & a_{12} & \cdots \\ a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix}\begin{bmatrix} 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & \cdots \\ 0 & -1 & \cdots \\ \vdots & \vdots & \end{bmatrix}\begin{bmatrix} a_{11} & -a_{12} & \cdots \\ a_{21} & -a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} = \begin{bmatrix} a_{11} & -a_{12} & \cdots \\ -a_{21} & a_{22} & \cdots \\ \vdots & \vdots & \end{bmatrix} = B.$$

Alternative

For any diagonal matrix D,

$$b_{ij} = d_i a_{ij} d_j.$$

As $b_{ij} = (-1)^{i+j}a_{ij}$, we have to choose D in such a way that $(-1)^{i+j} = d_i d_j$. This leads to the solution mentioned above.

(b) Because $\det D = \pm 1$, it follows that

$$\det B = \det\bigl(ADA\bigr) = \det D \cdot \det A \cdot \det D = \det A \cdot (\det D)^2 = \det A.$$

8.21 Because $\det A = 1$, Theorem 5 implies that the matrix A is invertible. In view of Cramer's Rule (Theorem 8), the entry s_j of the unique solution $\underline{s}$ of the system $A\underline{x} = \underline{b}$ is

$$\frac{\det A_j(\underline{b})}{\det A} = \det A_j(\underline{b}).$$

Since the coefficients of the matrix A and the entries of the vector $\underline{b}$ are integers, $\det A_j(\underline{b})$ is an integer too. So all the entries of the solution vector $\underline{s}$ are integers.